BENDING VIBRATIONAL DATA ACCURACY STUDY

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By

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Prepared For

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Alabama 35812
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FOREWORD

This project was sponsored by the George C. Marshall Space Flight Center under NASA Contract NAS8-25458. The work was performed under the technical direction of Larry A. Kiefling, MSFC Code S & E-Aero-DDS.

We wish to express our appreciation to Mr. Kiefling for his thoughtful contributions to the partial derivative development.

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1.0 INTRODUCTION AND SUMMARY

This report presents the theory and describes the operation of a computer program designed to predict uncertainty in structural modal characteristics based on uncertainty in structural physical properties. The program, entitled VIDAP (Vibrational Data Accuracy Program), can handle both stiffness and mass uncertainty and can work with an arbitrary stiffness matrix or one which involves beam or plate elements. The program and the supporting theory have the following features:

- a linear statistical model which can accurately predict uncertainties of selected frequencies and modes based on the uncertainty in properties of individual elements.
- the program never handles matrices of dimension larger than the total degrees of freedom of the system. The program can handle problems of up to 300 degrees of freedom.
- the computation speed of the program is less than that required for computation of eigenvectors and eigenvalues.
- the program stands alone from any structural dynamics program, requiring only the eigenvalues and eigenvectors, the mass and stiffness matrices, and certain element properties as input.
- the input procedure is of such a form that the user need not have any knowledge of statistics in order to get an acceptable answer.

Included with the description of the program and theory are two examples which demonstrate the operation. The results from the two examples, a four degree-of-freedom longitudinal rod and an S II longitudinal vibration model, are compared with Monte Carlo or other independent solutions to confirm the accuracy of the VIDAP solution. Conclusions from these examples are:

- The VIDAP theory is substantiated by excellent correlation of results from the four degree-of-freedom system.
- The VIDAP program can compute eigenvalue statistics very accurately in any size model but has difficulty with the accurate prediction of the eigenvector statistics in large models. The difficulties apparently lie with

roundoff or nonlinearity of the eigenvector components used in the development of the partial derivatives. The difficulties may be attributable to the particular test problem but may exist in many other untried problems as well. One of the recommendations of the study is to perform further work on the influence of ill conditioning and roundoff upon the eigenvector statistics.

A user's manual is included in Appendix B. This appendix, supported by the theoretical development should make the report self-contained in providing adequate information for the operation of VIDAP.

2.0 PROCEDURE

2.1 Statistical Background

Before developing the model for computing the frequency and modal statistics, it is advisable to review the statistical concepts which are used throughout. A simple linear relationship between the vector {x} and vector {y} is presented in Equation (2-1).

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \tag{2-1}$$

$$\{x\} = [A]\ \{y\}$$

If the components of {y} are random, then x_1, x_2, and x_3 are random as well. To compute the variances of x_1, x_2 and x_3 it is necessary to perform a matrix manipulation with [A] and a covariance matrix of {y} which is described in Equation (2-2).

$$\text{y covariance matrix} = [\Sigma y] = \begin{bmatrix} \sigma^2 y_1 & \sigma y_1 y_2 & \sigma y_1 y_3 \\ \sigma y_2 y_1 & \sigma^2 y_2 & \sigma y_2 y_3 \\ \sigma y_3 y_1 & \sigma y_3 y_2 & \sigma^2 y_3 \end{bmatrix} \tag{2-2}$$

The diagonal elements of the y covariance matrix, [Σy], are the standard deviations squared (the variances) of each of the components of {y}. The off-diagonal elements of the covariance matrix show how y_1 and y_2, for instance, are statistically correlated. If we assume that each of the components of {y} are statistically independent, then [Σy] becomes a diagonal matrix with off-diagonal elements equal to zero. However, statistical independence of the elements of {y} does not necessarily mean statistical independence of the elements of {x}.

The covariance matrix for the components of {x} is now written as a function of [A] and $[\Sigma_y]$ as shown in Equation (2-3).

$$[\Sigma_x] = [A]\ [\Sigma_y]\ [A]' \tag{2-3}$$

The x covariance matrix $[\Sigma_x]$ now has variances of the components of {x} along the diagonal and covariances of the components off the diagonal. To compute the correlation coefficients use the formula

$$\rho_{12} = \frac{\sigma_{x_1 x_2}}{\sigma_{x_1}\sigma_{x_2}} \tag{2-4}$$

where ρ_{12} is the correlation coefficient for elements x_1 and x_2, $\sigma_{x_1 x_2}$ is the covariance (from $[\Sigma_x]$) and σ_{x_1} and σ_{x_2} are the standard deviations.

The presentation above is, in essence, most of the statistical background necessary for the development of the covariance matrix for frequencies and modes.

2.2 Considerations in the Dynamic Model

Before developing the statistical model, let us review the general steps* involved in developing system stiffness and mass matrices. Consider as an example the truss on the next page.

* These steps are shown to clarify steps which will be used in the statistical development and need not represent any particular structural dynamics program.

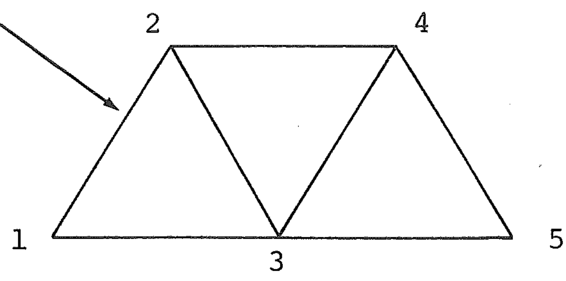

Figure 1. Truss Example

The truss is made up of seven elements. We will use Beam Element i as our example. When entering data into the program the user will identify that points 1 and 2 of the structure are connected by a beam with a possible twelve degrees of freedom (six at each end). The program has built into it a stiffness matrix format for a beam, so that when physical properties (E, I, A, L, etc.) are provided, the program will develop a stiffness matrix and a mass matrix for the beam in its own local coordinate system. To make these two matrices compatible with the global coordinate system, they are pre and post-multiplied by a rotation matrix whose elements are based on the orientation of Beam Element i relative to the global coordinates. The equation is shown below where $[{}^{(i)}K]$* is the stiffness matrix for beam element i in local coordinates, $[{}^{(i)}R]$ is the rotation matrix, and $[{}^{(i)}K_r]$ is the stiffness matrix in global coordinates.

$$[{}^{(i)}K_r] = [{}^{(i)}R]\,[{}^{(i)}K]\,[{}^{(i)}R]' \qquad (2\text{-}5)$$

At this point, the procedure used for Beam Element i is applied to every other element of the truss until seven independent pairs of stiffness and mass matrices have been constructed.

The stiffness matrix $[{}^{(i)}K_r]$ is associated with two nodes (1 and 2) in the global coordinate system. This is demonstrated on the next page.

* The pre-superscript $^{(i)}$ denotes association with the ith component in the structure. This nomenclature will is used throughout the report.

$$[{}^{(i)}K_r] = \begin{bmatrix} [{}^{(i)}1\text{-}1] & [{}^{(i)}1\text{-}2] \\ [{}^{(i)}2\text{-}1] & [{}^{(i)}2\text{-}2] \end{bmatrix} \tag{2-6}$$

where the matrices identified by hyphenated terms represent 6X6 stiffness matrices related to the nodes ($[{}^{(i)}1\text{-}1]$ and $[{}^{(i)}2\text{-}2]$) or to the coupling between nodes ($[{}^{(i)}1\text{-}2]$ and $[{}^{(i)}2\text{-}1]$).

There is more than one way to develop the system stiffness matrix, but the method used here is based on the development of the nodal stiffness. Hence, to begin, a matrix is defined which is partitioned according to the nodes. No coordinates are deleted because of constraints.

$$\begin{array}{ccccccc} & \text{Node 1} & \text{Node 2} & \text{Node 3} & & & \\ [K]_{\text{undeleted}} = & \left[\begin{matrix} [1\text{-}1] \\ [2\text{-}1] \\ [3\text{-}1] \\ \cdot \end{matrix}\right. & \begin{matrix} [1\text{-}2] \\ [2\text{-}2] \\ [3\text{-}2] \\ \cdot \end{matrix} & \begin{matrix} [1\text{-}3] \\ [2\text{-}3] \\ [3\text{-}3] \\ \cdot \end{matrix} & \left.\begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right] & \begin{matrix} 6 \text{ d.o.f.} \\ \\ \\ \\ \end{matrix} & \begin{matrix} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \\ \\ \end{matrix} \end{array} \tag{2-7}$$

The matrix [1-1] is the sum of the $[{}^{(i)}1\text{-}1]$ nodal stiffness matrices identified in the stiffness matrices for the components, i.e. identified from partitioning $[{}^{(i)}K_r]$.

The following two steps are used to obtain the stiffness matrix for the constrained system; (1) identify the constraints for each node, (2) remove the columns and rows of the undeleted [K] corresponding to the constraints in the nodes. [K] is now reduced to the size corresponding to the number of dynamic degrees of freedom and is ready for eigenvalue/vector computation.

The path from element properties to system stiffness has now been completed. It is this path that is used in developing the linear statistical model.

2.3 The Statistical Development

The development of the linear statistical package is based on the assumption that a small perturbation of a structural element property causes small perturbations in the natural frequencies of the system and that the relationships are linear for the range of the variations considered. The equation for one frequency and one property can be expressed most simply as

$$d\omega = \frac{\partial \omega}{\partial p}\, dp \qquad (2\text{-}8)$$

where ω is a single frequency and p is some property of some element in the system. The term, $\partial \omega / \partial p$, is composed of all of the modifications to p which take place when tracing through the system from the property, through the elemental stiffness matrix, through a rotation, through a compatibility matrix and finally through the eigenvalue computation. We can show this symbolically in a series of partial derivatives

The effect of rotating from local to global coordinates

The dependence of the component stiffness matrix on the property p

$$d\omega = \left(\frac{\partial \omega}{\partial \lambda}\right) \left(\frac{\partial \lambda}{\partial k_{syst}}\right) \left(\frac{\partial k_r}{\partial k}\right) \left(\frac{\partial k}{\partial p}\right) dp. \qquad (2\text{-}9)$$

The dependence of an eigenvalue upon the variation of an element in the stiffness matrix

The first two partial derivatives were developed in Reference (1). They are

$$\frac{\partial \omega_i}{\partial \lambda_i} = \frac{1}{2\sqrt{\lambda_i}} \qquad (2\text{-}10)$$

and

$$\frac{\partial \lambda_i}{\partial k_{pq}} = \frac{x_{pi} x_{qi}}{\{x_i\}'[M]\{x_i\}} \qquad (2\text{-}11)$$

where k_{pq} is the pq^{th} element in the system stiffness matrix, and x_{pi} and x_{qi} are the p^{th} and q^{th} elements in the i^{th} eigenvector.

Similar expressions were derived for $\frac{\partial \lambda_i}{\partial m_{pq}}$, $\frac{\partial x_{ji}}{\partial k_{pq}}$, and $\frac{\partial x_{ji}}{\partial m_{pq}}$, although the expressions concerning the eigenvector components are rederived in a more convenient form in Section 3 of this report.

Of the partial derivative expressions shown in (2-9), all but the first are matrices or vectors rather than scalars; and if the number of frequencies and elemental properties are increased and if modes are considered, all become nonscalar (i.e. matrix) expressions. At this point, we will attempt to define each of these expressions more exactly starting with the physical properties and working to the system stiffness matrix.

Using a beam element such as that discussed in Section 2.2, develop a vector of the physical properties. These properties would normally be included in the program input data.

$$\{p\} = \begin{Bmatrix} E \\ I \\ A \\ \nu \\ \cdot \\ \cdot \end{Bmatrix} \tag{2-12}$$

By perturbing each of the properties, linear expressions can be written for relating the components of the elemental stiffness matrix to the properties. The expressions can be put into the following matrix form:

$$\left\{ {}^{(i)}dk \right\} = \begin{Bmatrix} {}^{(i)}dk_{11} \\ dk_{12} \\ dk_{13} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \left[\frac{{}^{(i)}\partial (k)}{\partial (p)} \right] \left\{ {}^{(i)}dp \right\} \tag{2-13}$$

It is, however, more convenient to keep the partial derivatives, $\frac{\partial k_{ij}}{\partial p_\ell}$, in a matrix form until the stiffness elements have been rotated into system or global coordinates. Hence the stiffness matrix is differentiated n times, once for each property which can vary. There are now n matrices of partial derivatives as shown below containing all the sensitivities of the stiffness elements to the properties

$$\left[{}^{(i)}\frac{\partial(k)}{\partial p_1} \right], \quad \left[{}^{(i)}\frac{\partial(k)}{\partial p_2} \right], \quad . \ . \ . \ , \left[{}^{(i)}\frac{\partial(k)}{\partial p_n} \right]$$

To put these partial derivatives into system coordinates pre and post multiply these matrices by the rotation matrix $[{}^{(i)}R]$ for element i.

$$\left[{}^{(i)}\frac{\partial(k)_r}{\partial p_1} \right] = \left[{}^{(i)}R \right] \left[{}^{(i)}\frac{\partial(k)}{\partial p_1} \right] \left[{}^{(i)}R \right]'$$

$$\left[{}^{(i)}\frac{\partial(k)_r}{\partial p_2} \right] = \left[{}^{(i)}R \right] \left[{}^{(i)}\frac{\partial(k)}{\partial p_2} \right] \left[{}^{(i)}R \right]' \qquad (2\text{-}14)$$

$$\vdots \qquad\qquad\qquad \vdots$$

The elements of $\left[{}^{(i)}\frac{\partial(k)_r}{\partial p_i} \right]$, etc. can now be removed and placed into columns with each column representing a dependency upon a different property, p.

$$\left[{}^{(i)}\frac{\partial(k)_r}{\partial(p)} \right] = \begin{bmatrix} \frac{\partial k_{11_r}}{\partial p_1} & \frac{\partial k_{11_r}}{\partial p_2} & \frac{\partial k_{11_r}}{\partial p_3} & \cdots \\ \frac{\partial k_{12_r}}{\partial p_1} & \frac{\partial k_{12_r}}{\partial p_2} & \frac{\partial k_{12_r}}{\partial p_3} & \cdots \\ \frac{\partial k_{13_r}}{\partial p_1} & \frac{\partial k_{13_r}}{\partial p_2} & \frac{\partial k_{13_r}}{\partial p_3} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix} \qquad (2\text{-}15)$$

Note that $\left[\frac{{}^{(i)}\partial (k)_r}{\partial (p)} \right]$ is a matrix of n columns representing the n properties and m rows representing the m elements in $\left[{}^{(i)}k_r \right]$.

This matrix (Equation (2-15)) encompasses two of the partial derivatives shown in Equation (2-9), $\left[\frac{\partial k_r}{\partial k} \right]$ and $\left[\frac{\partial k}{\partial p} \right]$.

The components of the matrix $\left[\frac{{}^{(i)}\partial (k)_r}{\partial (p)} \right]$ can be predeveloped within the computer program because they are based on the same fixed expressions used to develop the stiffness matrix.

This generally covers the development of partial derivatives relating changes in physical properties to changes in modal characteristics. One additional point is worth mentioning, however. In developing the partial derivatives $\frac{\partial \lambda_i}{\partial k_{pq}}$ and $\frac{\partial x_{ji}}{\partial k_{pq}}$, note that not all of the partial derivatives are required; only those with respect to system stiffness elements k_{rs} which correspond to elements ${}^{(i)}k_{rs}$ in the rotated elemental stiffness matrix. Hence, logic must be introduced into the program to compute only those partial derivatives which relate to structural element i. We can denote this consideration for element i by introducing the presuperscript (i) into the matrix expression $\left[\frac{\partial (\lambda)}{\partial (k_{syst})} \right]$.

The expression in Equation (2-9) can be written as a product of a series of matrices (it is convenient to work with $d\lambda$ rather than $d\omega$; results can be put in terms of ω in the final operation).

$$\left\{ {}^{(i)}d\lambda \right\} = \left[\frac{{}^{(i)}\partial (\lambda)}{\partial (k_{syst})} \right] \left[\frac{{}^{(i)}\partial (k)_r}{\partial (p)} \right] \left\{ {}^{(i)}dp \right\}^{*} \tag{2-16}$$

and

$$\left\{ {}^{(i)}dx \right\} = \left[\frac{{}^{(i)}\partial (x)}{\partial (k_{syst})} \right] \left[\frac{{}^{(i)}\partial (k)_r}{\partial (p)} \right] \left\{ {}^{(i)}dp \right\} \tag{2-17}$$

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* The development here is for stiffness uncertainty alone. The method is the same for mass, and mass uncertainty is included in the VIDAP program. See Sections 4 and 5 for further details.

For convenience, rewrite as

$$\left\{ {}^{(i)}d\lambda \right\} = [B][C]\left\{ {}^{(i)}dp \right\} \tag{2-18}$$

$$\left\{ {}^{(i)}dx \right\} = [D][C]\left\{ {}^{(i)}dp \right\} \tag{2-19}$$

Thus, we have complete linear expressions showing the dependence of the eigenvalues and modes upon the physical properties of Beam Element i. Now, using Equation (2-3), write directly the covariance matrices for the eigenvalues and the eigenvectors.

$$\left[{}^{(i)}\Sigma_\lambda \right] = [B][C]\left[{}^{(i)}\Sigma_p \right][C]'[B]' \tag{2-20}$$

$$\left[{}^{(i)}\Sigma_x \right] = [D][C]\left[{}^{(i)}\Sigma_p \right][C]'[D]' \tag{2-21}$$

If more than one element in the structure has random properties, the process to produce $[{}^{(i)}\Sigma_\lambda]$ and $[{}^{(i)}\Sigma_x]$ can be repeated and the results combined to form the final covariance matrices for the eigenvalues and eigenvectors. Incidently, it is this reversal of the combination procedure and the assumption of statistical independence from structural element to structural element that permits the use of much smaller matrices compared to the method presented in Reference 1.

If the user is interested in a point on the structure between nodes, an additional computational procedure must be added to properly account for the variances and covariances of the modes.

2.4 Implementation of the Statistical Model

All the partial derivative expressions presented in (2-16) and (2-17) are developed within the computer program since they are based on properties or expressions already being considered in the development of the stiffness and mass matrix. The program requires only the system mass and stiffness matrices, the eigenvalues and eigenvectors, details of the elements with random properties, statistics of the properties, and constraints.

The second aspect of the problem is the size of the matrices involved and the kinds of manipulation. The procedure required to obtain $\frac{\partial (x)}{\partial (k,m)}$ always requires the inversion (or simultaneous equation solution) of $(n-1)x(n-1)$ matrices making this the most expensive part of the statistical computations. The other processes are primarily the development of elements by simple formulae or the multiplication of matrices, both of which are considerably cheaper than inversion. Every effort has been made to minimize the matrix storage requirements by using only half of the symmetric matrices and storing in a new matrix having the number of columns corresponding to the semi-bandwidth and the number of rows corresponding to the degrees of freedom. The simultaneous equation solution for $\frac{\partial x_{ji}}{\partial k_{pq}}$ (to be shown in Section 3) is accomplished, in part, by triangular decomposition of the matrix expression $[K-\lambda_i M]$. This is the fastest and most accurate method available and requires the least storage.

The output of the statistical data can be quite voluminous. For instance, in displaying the statistical parameters for all 100 eigenvectors of a 100 degree-of-freedom system, the covariance matrix has the dimensions of 10,000 x 10,000. Since the user will never have need for all of this data, he can confine the output to the specific eigenvectors and sections of the eigenvectors which are of interest.

3.0 EIGENVALUE AND EIGENVECTOR PARTIAL DERIVATIVES

3.1 Eigenvalue Partial Derivatives

The eigenvalue equation to be treated is

$$Kx_i = \lambda_i M x_i \tag{3-1}$$

where K and M are symmetric stiffness and mass matrices, x_i is a column vector of displacements (the ith eigenvector) and λ_i is a scalar (the ith eigenvalue). Considering each term to be a variable, differentiate and premultiply by the transpose x_j' to obtain

$$x_j'(K - \lambda_i M)dx_i = d\lambda_i x_j' M x_i - x_j' dK x_i + \lambda_i x_j' dM x_i \tag{3-2}$$

Since $x_j'K = \lambda_j x_j' M$, Equation (3-2) reduces to

$$(\lambda_j - \lambda_i)x_j' M dx_i = d\lambda_i x_j' M x_i - x_j' dK x_i + \lambda_i x_j' dM x_i \tag{3-3}$$

When j = i, the left side of Equation (3-3) is zero and we obtain

$$d\lambda_i = \frac{x_i' dK x_i}{x_i' M x_i} - \lambda_i \frac{x_i' dM x_i}{x_i' M x_i} \tag{3-4}$$

The product $x_i' dK x_i$ is scalar and can be expressed in terms of a double summation

$$x_i' dK x_i = \sum_{r=1}^{n} \sum_{s=1}^{n} x_{ri} dk_{rs} x_{si} \tag{3-5}$$

where x_{ri} and x_{si} are the rth and sth elements of the eigenvector x_i and dk_{rs} is the rs element in the matrix dK.

Substituting Eq. (3-5) and a similar expression for $x_j' dMx_i$ into Eq. (3-4), we find

$$d\lambda_i = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{x_{ri} x_{si}}{x_i' M x_i} dk_{rs} - \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\lambda_i x_{ri} x_{si}}{x_i' M x_i} dm_{rs} \tag{3-6}$$

Equation (3-6) is equivalent to a chain of partial derivatives

$$d\lambda_i = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_i}{\partial k_{rs}} dk_{rs} + \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial \lambda_i}{\partial m_{rs}} dm_{rs} \tag{3-7}$$

where
$$\frac{\partial \lambda_i}{\partial k_{rs}} = \frac{x_{ri} x_{si}}{x_i' M x_i} \tag{3-8}$$

$$\frac{\partial \lambda_i}{\partial m_{rs}} = -\lambda_i \frac{x_{ri} x_{si}}{x_i M x_i} \tag{3-9}$$

3.2 Eigenvector Partial Derivatives

Differentiate Eq. (3-1),

$$[K - \lambda_i M]dx_i = d\lambda_i Mx_i - dKx_i + \lambda_i dMx_i \tag{3-10}$$

and substitute Eq. (3-4) for $d\lambda_i$

$$[K - \lambda_i M]dx_i = \left(\frac{x_i' dKx_i}{x_i' Mx_i} - \lambda_i \frac{x_i' dMx_i}{x_i' Mx_i} \right) Mx_i - dKx_i + \lambda_i dMx_i \tag{3-11}$$

Choose for example a dependence upon stiffness element k_{rs}. Then

$$[K - \lambda_i M]dx_i = \frac{x_{ri} x_{si} dk_{rs}}{x_i' Mx_i} Mx_i - dk_{rs}\, x_{si}\{\delta_{jr}\} \tag{3-12}$$

where the expression $x_{si}\{\delta_{jr}\}$ is equivalent to a zero vector with a non-zero element, x_{si}, in the rth row. δ_{jr} is a Kronecker delta defined by

$$\delta_{jr} = 0, \quad r \neq j$$

$$\delta_{jr} = 1, \quad r = j$$

j represents the number of the row or element in the vector in this case.

$$[K - \lambda_i M]dx_i = \left(\frac{x_{ri} x_{si}}{x_i' Mx_i} Mx_i - x_{si}\{\delta_{jr}\} \right) dk_{rs}$$

or equivalently

$$[K - \lambda_i M]dx_i = \left(\frac{x_{ri} x_{si}}{x_i' Mx_i} Mx_i - \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ x_{si} \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \right) dk_{rs} \tag{3-13}$$

Let $F_i = K - \lambda_i M$. If λ_i is a single root (eigenvalue) then F_i is of rank n-1 and cannot be inverted. Therefore no unique solution exists for the vector dx_i as shown in Eq. (3-13). A solution can be obtained however if one of the elements of the vector dx_i is fixed equal to zero and any n-1 equations of F_i are used to solve for the remaining elements of dx_i.* This yields a solution for dx_i which is dependent on the fixed element. It is convenient to omit the equation (identified by row in F_i) which corresponds to the number of the element in the vector dx_i. For instance if the first element in dx_i (i.e. dx_{1i}) is fixed, then the first row in F_i would be removed and the first element in the vector

$$\left(\frac{x_{ri}x_{si}}{\{x_i\}'[M]\{x_i\}} [M]\{x_i\} - x_{si}\{\delta_{jr}\} \right) dk_{rs}$$

which forms the right-hand side of Equation (3-13) is removed. This omission of row corresponding to the zeroed dx_i element maintains the symmetry of the reduced form of F_i and simplifies the problem solution.

Let us now introduce the notation $[\ \overline{F_i}^u\ ]$ where the bar and superscript u represent the removal of the uth row and column of $[\ F_i\ ]$. The new matrix is of rank n-1 and hence can be inverted to solve for $\left\{\overline{dx_i}^u\right\}$ which is a vector of dimension n-1 where dx_{ui} has been set equal to zero and removed. Equation (3-13) now becomes

$$[\ \overline{F_i}^u\ ] \left\{\overline{dx_i}^u\right\} = \left(\frac{x_{ri}x_{si}}{x_i'Mx_i} \left\{\overline{Mx_i}^u\right\} - x_{si} \left\{\overline{\delta_{jr}}^u\right\} \right) dk_{rs} \qquad (3\text{-}14)$$

Similarly for dm_{rs} we have

$$[\ \overline{F_i}^u\ ] \left\{\overline{dx_i}^u\right\} = \lambda_i \left(\frac{x_{ri}x_{si}}{x_i'Mx_i} \left\{\overline{Mx_i}^u\right\} - x_{si} \left\{\overline{\delta_{jr}}^u\right\} \right) dm_{rs} \qquad (3\text{-}15)$$

Completing the solution for the partial derivatives

$$\left\{\overline{\frac{\partial x_i}{\partial k_{rs}}}^u\right\} = [\ \overline{F_i}^u\ ]^{-1} \left(\frac{x_{ri}x_{si}}{x_i'Mx_i} \left\{\overline{Mx_i}^u\right\} - x_{si} \left\{\overline{\delta_{jr}}^u\right\} \right) \qquad (3\text{-}16)$$

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* This approach and development is based on the note, "Comment on 'The Eigenvalue Problem for Structural Systems with Statistical Properties'", by Larry A. Kiefling and published in the <u>AIAA Journal</u>, July 1970.

and $\left\{\dfrac{\partial \bar{x}_i^{u}}{\partial m_{rs}}\right\} = -\lambda_i \left\{\dfrac{\partial \bar{x}_i^{u}}{\partial k_{rs}}\right\}$ (3-17)

Numerically it is not necessary to invert $\left[\bar{F}_i^{u}\right]$ since Equation (3-16) can be solved as a set of simultaneous algebraic equations. A discussion of the matrix decomposition procedure selected to solve for $\left\{\dfrac{\partial \bar{x}_i^{u}}{\partial k_{rs}}\right\}$ is presented in Section 6.3.

The eigenvector partial derivatives in Equation (3-16) are not truly representative of any system unless a restriction has been placed upon the eigenvectors that the element x_{ui} of each eigenvector x_i be held constant. In instances where the eigenvectors are normalized such that the first element is always one, Equation (3-16) would be valid for the superscript u equal to one. That is to say, $\dfrac{\partial x_{1i}}{\partial k_{rs}} = 0$ and x_{1i} = constant. If, however, the eigenvector solution requires a constant generalized mass ($x_i^T M x_i$ = const.), Equation (3-16) is not satisfactory and a further operation is necessary to obtain the full vector of partial derivatives, $\left\{\dfrac{\partial x_i}{\partial k_{rs}}\right\}$.

Define $\left\{\dfrac{\partial x_i^{u}}{\partial k_{rs}}\right\}$ as the vector of partial derivatives developed from (3-16) but with a zero inserted in the uth element rather than having the uth element omitted. That is

$$\left\{\frac{\partial x_i^{u}}{\partial k_{rs}}\right\} = \left\{\begin{matrix} \dfrac{\partial x_{1i}}{\partial k_{rs}} \\[2ex] \dfrac{\partial x_{2i}}{\partial k_{rs}} \\ \cdot \\ \cdot \\ \dfrac{\partial x_{(u-1)i}}{\partial k_{rs}} \\[2ex] 0 \\ \dfrac{\partial x_{(u-1)i}}{\partial k_{rs}} \\ \cdot \\ \cdot \end{matrix}\right\} \qquad (3\text{-}18)$$

Next form the generalized mass using $\left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\}$

$$\left(\{x_i\} + \left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\} dk_{rs}\right)' [M] \left(\{x_i\} + \left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\} dk_{rs}\right) = M_{irs}^u \tag{3-19}$$

whereas M_i as defined below is the generalized mass without perturbation.

$$\{x_i\}'[M]\{x_i\} = M_i \tag{3-20}$$

The objective is to find a new vector $\left\{\frac{\partial x_i}{\partial k_{rs}}\right\}$ with no fixed elements, which will keep the generalized mass M_i constant.

$$\left(\{x_i\} + \left\{\frac{\partial x_i}{\partial k_{rs}}\right\} dk_{rs}\right)' [M] \left(\{x_i\} + \left\{\frac{\partial x_i}{\partial k_{rs}}\right\} dk_{rs}\right) = M_i \tag{3-21}$$

Divide Equations (3-19) and (3-21) through by M_{irs}^u and M_i and equate.

$$\frac{1}{M_{irs}^u} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(x_{pi} + \frac{\partial x_{pi}^u}{\partial k_{rs}} dk_{rs} \right) m_{pq} \left(x_{qi} + \frac{\partial x_{qi}^u}{\partial k_{rs}} dk_{rs} \right)$$

$$= \frac{1}{M_i} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(x_{pi} + \frac{\partial x_{pi}}{\partial k_{rs}} dk_{rs} \right) m_{pq} \left(x_{qi} + \frac{\partial x_{qi}}{\partial k_{rs}} dk_{rs} \right) \tag{3-22}$$

From Equation (3-22) it is possible to equate eigenvector elements, hence

$$x_{pi} + \frac{\partial x_{pi}}{\partial k_{rs}} dk_{rs} = \sqrt{\frac{M_i}{M_{irs}^u}} \left(x_{pi} + \frac{\partial x_{pi}^u}{\partial k_{rs}} dk_{rs} \right)$$

$$\frac{\partial x_{pi}}{\partial k_{rs}} dk_{rs} = \left(\sqrt{\frac{M_i}{M_{irs}^u}} - 1 \right) x_{pi} + \sqrt{\frac{M_i}{M_{irs}^u}} \frac{\partial x_{pi}^u}{\partial x_{rs}} dk_{rs} \tag{3-23}$$

The ratio $\sqrt{\dfrac{M_i}{M_{irs}^u}}$ contains the derivative dk_{rs}. Multiplying out M_{irs}^u from Equation (3-19), we have

$$M_{irs}^u = \{x_i\}'[M]\{x_i\} + \left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\}'[M]\{x_i\}dk_{rs}$$

$$+ \{x_i\}'[M]\left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\} dk_{rs}$$

$$+\left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\}'[M]\left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\}\left(dk_{rs}\right)^2 \tag{3-24}$$

Note that [M] is symmetrical and $(dk_{rs})^2 << dk_{rs}$. Equation (3-24) becomes

$$M_{irs}^u = M_i + 2\left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\}'[M]\{x_i\}dk_{rs} \tag{3-25}$$

Then

$$\sqrt{\frac{M_i}{M_{irs}^u}} = \left(\frac{1}{1 + \dfrac{2}{M_i}\left\{\dfrac{\partial x_i^u}{\partial k_{rs}}\right\}'[M]\{x_i\}dk_{rs}}\right)^{\frac{1}{2}} \tag{3-26}$$

which expands into

$$\sqrt{\frac{M_i}{M_{irs}^u}} = 1 - \frac{1}{2}\left(\frac{2}{M_i}\left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\}'[M]\{x_i\}dk_{rs}\right)$$

$$+ \frac{1\cdot 3}{2\cdot 4}\left(\frac{2}{M_i}\left\{\frac{\partial x_i^u}{\partial k_{rs}}\right\}'[M]\{x_i\}dk_{rs}\right)^2$$

$$\cdots \tag{3-27}$$

Again assume $(dk_{rs})^2 << dk_{rs}$, causing the higher order terms of (3-27) to vanish.

Next, substitute this linearized form of (3-27) into (3-23) to get

$$\frac{\partial x_{pi}}{\partial k_{rs}} dk_{rs} = -\frac{x_{pi}}{M_i}\left\{\frac{\partial x_i}{\partial k_{rs}}^u\right\}'[M]\{x_i\}dk_{rs}$$

$$+\left(1 - \frac{1}{M_i}\left\{\frac{\partial x_i}{\partial k_{rs}}^u\right\}'[M]\{x_i\}dk_{rs}\right)\frac{\partial x_{pi}}{\partial k_{rs}}^u dk_{rs}$$

$$\cong \left(\frac{\partial x_{pi}}{\partial k_{rs}}^u - \frac{x_{pi}}{M_i}\left\{\frac{\partial x_i}{\partial k_{rs}}^u\right\}'[M]\{x_i\}\right)dk_{rs} \tag{3-28}$$

and

$$\frac{\partial x_{pi}}{\partial k_{rs}} = \frac{\partial x_{pi}}{\partial k_{rs}}^u - \frac{x_{pi}}{M_i}\left\{\frac{\partial x_i}{\partial k_{rs}}^u\right\}'[M]\{x_i\} \tag{3-29}$$

Note in Equation (3-29) that $\frac{\partial x_{pi}}{\partial k_{rs}}$ is not symmetrical, i.e. $\frac{\partial x_{pi}}{\partial k_{rs}} \neq \frac{\partial x_{pi}}{\partial k_{sr}}$.

This means that the property of symmetry which is so convenient in the handling of the mass and stiffness matrices is lost and the partial derivatives of x_{pi} must be computed with respect to every element of the stiffness matrix. However, since the stiffness matrix will always be symmetrical, there is no reason to treat the partial derivatives for symmetrical elements separately. Thus add the two and treat the sum. Using Equation (3-16),

$$\left\{\frac{\overline{\partial x_i}^u}{\partial k_{rs}}\right\} + \left\{\frac{\overline{\partial x_i}^u}{\partial k_{sr}}\right\}$$

$$= \left[\overline{F_i}^u\right]^{-1}\left(\frac{2x_{ri}x_{si}}{x_i'Mx_i}\left\{\overline{Mx_i}^u\right\} - x_{si}\left\{\overline{\delta_{jr}}^u\right\} - x_{ri}\left\{\overline{\delta_{js}}^u\right\}\right)$$

$$= \left[\overline{F_i}^u\right]^{-1}\left(\frac{2x_{ri}x_{si}}{x_i'Mx_i}\left\{\overline{Mx_i}^u\right\} - \left\{\overline{x_{si}\delta_{jr} + x_{ri}\delta_{js}}^u\right\}\right) \tag{3-30}$$

Equation (3-30) is very similar to (3-16) except for the factor of 2 and the additional element.

4.0 ELEMENT PROPERTY SENSITIVITY MATRICES

The structural makeup of a great many structures can be described by a series of beam and plate finite elements. In this study, a standard beam element (12 degrees of freedom) and a triangular plate element (15 degrees of freedom) were used in developing the relationship between physical properties of the structure and the eigenvalues and eigenvectors of the dynamic system. In addition, procedures are described and allowances are made for the inclusion of stiffness matrices for other element-types.

The procedure used for the development of the stiffness matrix partial derivatives is summarized as follows:

(1) the stiffness matrix is identified for the structural element of interest in local (element) coordinates

(2) variables in the stiffness matrix which can be random variables are identified (note structural geometry such as locations of nodes are not considered to be random).

(3) the stiffness matrix is differentiated with respect to each of the properties which can be random. Each set of partial derivatives (with respect to a property) is stored in a separate matrix.

(4) rotation matrices are developed from the coordinates of the nodes.

(5) the matrices of partial derivatives in local (element) coordinates are transformed into matrices of partial derivatives in system (global) coordinates.

(6) the partial derivatives from each of the matrices are removed and put in columnar form in a new matrix. Each column represents sensitivities of stiffness elements to a different physical property.

The procedure as described is used on both the beam and plate elements in the sections that follow.

4.1 Beam Element

4.1.1 Introduction

The beam element stiffness property sensitivity matrices with respect to a global coordinate system are developed in this section. The end result is a property sensitivity matrix which relates the partial derivatives of member geometric and material properties to the partial derivatives of each element in the beam stiffness matrix.

The beam element is assumed to be straight and have a uniform cross section. Material properties are invariant along the element's length and both internal shear and bending deformations are considered. Each cross-section is capable of resisting axial and shearing forces, bending moments about the two principal axes in the plane of the cross-section, and a twisting moment about the centroidal axis. Figure 4-1 shows a typical beam element with axes Y_m and Z_m corresponding to the principal axes of the cross-section and the centroidal axis, X_m. The six independent displacements at each end of the element are shown in this figure and noted as type $\{U\}$ displacements.

It is important to emphasize two points. First, the member centroidal axis, X_m, is always directed along the length of the element and corresponds to the bending neutral axis of the beam. Second, the Y_m and Z_m axes are identical with the two principal axes of the beam cross-section. The importance of this is the uncoupling of the induced stresses caused by the bending moments corresponding to U_5, U_6, U_{11} and U_{12} (see Reference 3, Page 70).

4.1.2 Beam Stiffness Property Sensitivity Matrix

The derivation of the beam stiffness matrix for the general beam element shown in Figure 4-1 can be readily available in a number of references (e.g., see Ref. 3, pgs. 70-82). Table 4-1 gives the resulting member stiffness matrix which was derived including shear and bending deformations. The beam properties that are present in this matrix, and the ones that may be considered as random variables are:

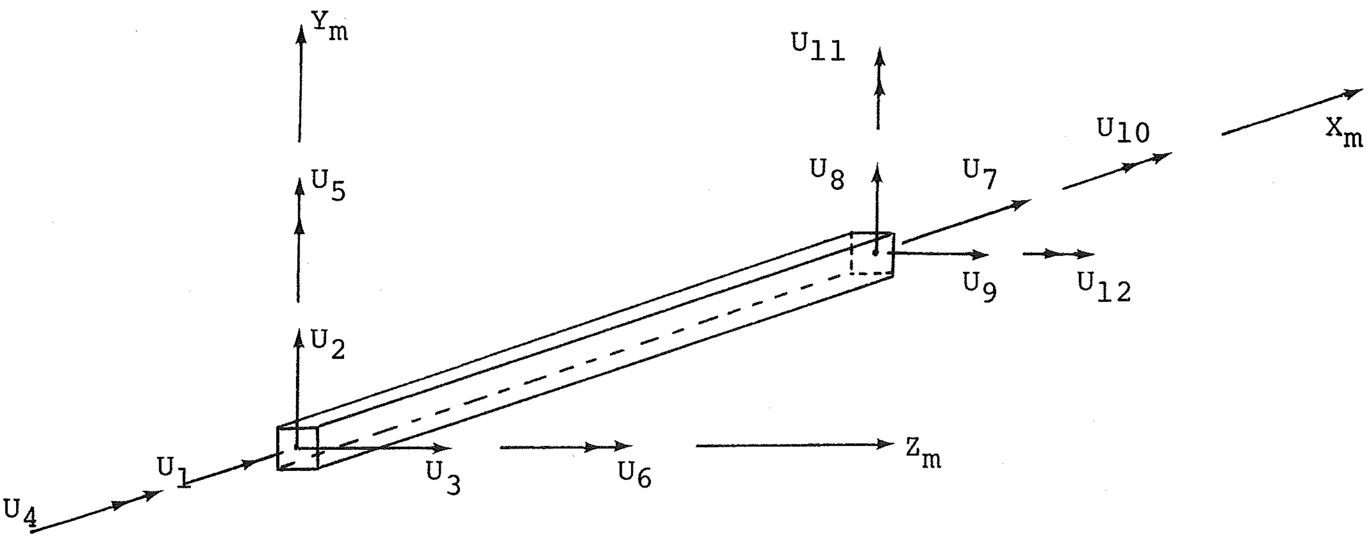


Figure 4-1 General Beam Finite Element

Table 4-1 Beam Element Stiffness Matrix, [K]

$$
\begin{bmatrix}
\frac{EA}{L} & & & & & & & & & & & \\
0 & \frac{12EI_3}{L^3(1+\Phi_2)} & & & & & & & & & & \\
0 & 0 & \frac{12EI_2}{L^3(1+\Phi_3)} & & & & & & & & & \\
0 & 0 & 0 & \frac{EJ}{2(1+\nu)L} & & & & \text{Symmetric} & & & & \\
0 & 0 & \frac{-6EI_2}{L^2(1+\Phi_3)} & 0 & \frac{(4+\Phi_3)EI_2}{L(1+\Phi_3)} & & & & & & & \\
0 & \frac{6EI_3}{L^2(1+\Phi_2)} & 0 & 0 & 0 & \frac{(4+\Phi_2)EI_3}{L(1+\Phi_2)} & & & & & & \\
-\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{L} & & & & & \\
0 & \frac{-12EI_3}{L^3(1+\Phi_2)} & 0 & 0 & 0 & \frac{-6EI_3}{L^2(1+\Phi_2)} & 0 & \frac{12EI_3}{L^3(1+\Phi_2)} & & & & \\
0 & 0 & \frac{-12EI_2}{L^3(1+\Phi_3)} & 0 & \frac{6EI_2}{L^2(1+\Phi_3)} & 0 & 0 & 0 & \frac{12EI_2}{L^3(1+\Phi_3)} & & & \\
0 & 0 & 0 & -\frac{EJ}{2(1+\nu)L} & 0 & 0 & 0 & 0 & 0 & \frac{EJ}{2(1+\nu)L} & & \\
0 & 0 & \frac{-6EI_2}{L^2(1+\Phi_3)} & 0 & \frac{(2-\Phi_3)EI_2}{L(1+\Phi_3)} & 0 & 0 & 0 & \frac{6EI_2}{L^2(1+\Phi_3)} & 0 & \frac{(4+\Phi_3)EI_2}{L(1+\Phi_3)} & \\
0 & \frac{6EI_3}{L^2(1+\Phi_2)} & 0 & 0 & 0 & \frac{(2-\Phi_2)EI_3}{L(1+\Phi_2)} & 0 & \frac{-6EI_3}{L^2(1+\Phi_2)} & 0 & 0 & 0 & \frac{(4+\Phi_2)EI_3}{L(1+\Phi_2)}
\end{bmatrix}
$$

$$\Phi_2 = \frac{12EI_3}{GASF_3L^2} = \frac{24\, I_3(1+\nu)}{ASF_3L^2}$$

$$\Phi_3 = \frac{12EI_2}{GASF_2L^2} = \frac{24I_2(1+\nu)}{ASF_2L^2}$$

where $G = \frac{E}{2(1+\nu)}$

E - Young's modulus of elasticity

ν - Poisson's Ratio

A - Beam cross-sectional area

I_2, I_3 - Beam cross-sectional moment of inertia about Y_m and Z_m axes, respectively.

J - Beam cross-sectional polar moment of inertia about X_m axis.

SF_3, SF_2 - Beam cross-sectional shear factors about the Y_m and Z_m axes, respectively.

It is noted that the properties which involve the geometric dimensions of the cross section are not independent random variables. In general, the covariances between these properties are non-zero and for each distinct cross-section (e.g. circular, square, tubular, etc.) there exists a unique relationship between the cross-section dimensional statistics and the above mentioned property statistics. This topic is discussed in detail in Section 5.

Central to the formulation of the natural frequency and modal statistics of a structure is the property sensitivity matrix. This matrix mathematically relates linear variations in the beam random properties to the force-displacement relationship (i.e. stiffness matrix). Corresponding to each beam property there is a unique property sensitivity matrix, noted

$$\frac{\partial [K_{beam}]}{\partial p} \; ; \; p = E, \nu, A, I_2, I_3, J, SF_2, SF_3$$

where

$[K_{beam}]$ = beam stiffness matrix in the X_m, Y_m, Z_m coordinate system

These eight sensitivity matrices are all of order 12 x 12 and each results from taking the partial derivative of each element in the stiffness matrix with respect to a prescribed property. For example,

$$\frac{\partial [K_{beam}]}{\partial J}$$

is a 12 x 12 property sensitivity matrix calculated by taking the partial derivative of each element in $[K_{beam}]$ with respect to J. Since only the elements in the fourth rows and columns contain J (Table 4-1) all other elements in the sensitivity matrix are zero. Therefore, the property sensitivity matrix corresponding to the torsional constant has only four non-zero elements - (4,4), (4,10), (10,4) and (10, 10). The other seven sensitivity matrices are, in general, more complicated, but the method of development is the same.

Certain characteristics of these property sensitivity matrices are worth noting. First, they are symmetric matrices because they are obtained by partial differentiation of the symmetric stiffness matrix. Second, they are not necessarily positive definite. This follows from observing that the terms on the diagonal of the property sensitivity matrices may be zero or negative.

Seven of the eight sensitivity matrices are shown in the following tables (Table 4-2 to 4-8). The elements are shown as a factor times the original stiffness element, e.g.

$$\frac{\partial k_{22}}{\partial A} = \frac{\Phi_2}{(1+\Phi_2)} \frac{k_{22}}{A}$$

The table for $\dfrac{\partial [K_{beam}]}{\partial E}$ is omitted because it is equivalent to $\frac{1}{E}[K_{beam}]$.

Table 4-2 Beam Element Sensitivity Matrix, $\dfrac{\partial\left[k_{beam}\right]}{\partial A}$

$$
\left[\begin{array}{cccccccccccc}
\frac{k_{11}}{A} & & & & & & & & & & & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{22}}{A} & & & & & & & & & & \\
0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{33}}{A} & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & \frac{3\Phi_3}{(1+\Phi_3)(4+\Phi_3)}\frac{k_{55}}{A} & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{53}}{A} & & & & \text{Symmetric} & & & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{62}}{A} & 0 & 0 & 0 & \frac{3\Phi_2}{(1+\Phi_2)(4+\Phi_2)}\frac{k_{66}}{A} & & & & & & \\
\frac{k_{71}}{A} & 0 & 0 & 0 & 0 & 0 & \frac{k_{77}}{A} & & & & & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{82}}{A} & 0 & 0 & 0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{86}}{A} & 0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{88}}{A} & & & & \\
0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{93}}{A} & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{95}}{A} & 0 & 0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{99}}{A} & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{11,3}}{A} & 0 & \frac{3\Phi_3}{(1+\Phi_3)(2-\Phi_3)}\frac{k_{11,5}}{A} & 0 & 0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{11,9}}{A} & 0 & \frac{3\Phi_3}{(1+\Phi_3)(4+\Phi_3)}\frac{k_{11,11}}{A} & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{12,2}}{A} & 0 & 0 & 0 & \frac{3\Phi_2}{(1+\Phi_2)(2-\Phi_2)}\frac{k_{12,6}}{A} & 0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{12,8}}{A} & 0 & 0 & 0 & \frac{3\Phi_2}{(1+\Phi_2)(4+\Phi_2)}\frac{k_{11,12}}{A}
\end{array}\right]
$$

Table 4-3 Beam Element Sensitivity Matrix, $\frac{\partial \lfloor K_{beam} \rfloor}{\partial I_2}$

$$
\begin{bmatrix}
0 & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & \\
0 & 0 & -\frac{1}{(1+\Phi_3)}\frac{k_{33}}{I_2} & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & -\frac{1}{(1+\Phi_3)}\frac{k_{53}}{I_2} & 0 & \frac{(\Phi_3^2+2\Phi_3+4)}{(1+\Phi_3)(4+\Phi_3)}\frac{k_{55}}{I_2} & & & & \text{Symmetric} & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & -\frac{1}{(1+\Phi_3)}\frac{k_{93}}{I_2} & 0 & -\frac{1}{(1+\Phi_3)}\frac{k_{95}}{I_2} & 0 & 0 & 0 & -\frac{1}{(1+\Phi_3)}\frac{k_{99}}{I_2} & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & -\frac{1}{(1+\Phi_3)}\frac{k_{11,3}}{I_2} & 0 & \frac{(2-2\Phi_3-\Phi_3^2)}{(1+\Phi_3)(2-\Phi_3)}\frac{k_{11,5}}{I_2} & 0 & 0 & 0 & -\frac{1}{(1+\Phi_3)}\frac{k_{11,9}}{I_2} & 0 & \frac{(\Phi_3^2+2\Phi_3+4)}{(1+\Phi_3)(4+\Phi_3)}\frac{k_{11,11}}{I_2} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Table 4-4 Beam Element Sensitivity Matrix, $\dfrac{\partial [k_{beam}]}{\partial I_3}$

$$
\begin{bmatrix}
0 & & & & & & & & & & & \\
0 & -\frac{1}{(1+\Phi_2)}\frac{k_{22}}{I_3} & & & & & & & & & & \\
0 & 0 & 0 & & & & & \text{Symmetric} & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & -\frac{1}{(1+\Phi_2)}\frac{k_{62}}{I_3} & 0 & 0 & 0 & \frac{(\Phi_2^2+2\Phi_2+4)}{(1+\Phi_2)(4+\Phi_2)}\frac{k_{66}}{I_3} & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & -\frac{1}{(1+\Phi_2)}\frac{k_{82}}{I_3} & 0 & 0 & 0 & -\frac{1}{(1+\Phi_2)}\frac{k_{86}}{I_3} & 0 & -\frac{1}{(1+\Phi_2)}\frac{k_{88}}{I_3} & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & -\frac{1}{(1+\Phi_2)}\frac{k_{12,2}}{I_3} & 0 & 0 & 0 & \frac{(2-2\Phi_2-\Phi_2^2)}{(1+\Phi_2)(2-\Phi_2)}\frac{k_{12,6}}{I_3} & 0 & -\frac{1}{(1+\Phi_2)}\frac{k_{12,8}}{I_3} & 0 & 0 & 0 & \frac{(\Phi_2^2+2\Phi_2+4)}{(1+\Phi_2)(4+\Phi_2)}\frac{k_{12,12}}{I_3}
\end{bmatrix}
$$

Table 4-5 [illegible]

$$
\left[\begin{array}{cccccccccccc}
0 & & & & & & & & & & & \\
0 & \frac{-\phi_2}{(1+\phi_2)}\frac{k_{22}}{(1+\nu)} & & & & & & & & & & \\
0 & 0 & \frac{-\phi_3}{(1+\phi_3)}\frac{k_{33}}{(1+\nu)} & & & & & & \text{Symmetric} & & & \\
0 & 0 & 0 & \frac{-k_{44}}{(1+\nu)} & & & & & & & & \\
0 & 0 & \frac{-\phi_3}{(1+\phi_3)}\frac{k_{53}}{(1+\nu)} & 0 & \frac{-3\phi_3 k_{55}}{(1+\phi_3)(4+\phi_3)(1+\nu)} & & & & & & & \\
0 & \frac{-\phi_2}{(1+\phi_2)}\frac{k_{62}}{(1+\nu)} & 0 & 0 & 0 & \frac{-3\phi_2 k_{66}}{(1+\phi_2)(4+\phi_2)(1+\nu)} & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & \frac{-\phi_2}{(1+\phi_2)}\frac{k_{82}}{(1+\nu)} & 0 & 0 & 0 & \frac{-\phi_2}{(1+\phi_2)}\frac{k_{86}}{(1+\nu)} & 0 & \frac{-\phi_2}{(1+\phi_2)}\frac{k_{88}}{(1+\nu)} & & & & \\
0 & 0 & \frac{-\phi_3}{(1+\phi_3)}\frac{k_{93}}{(1+\nu)} & 0 & \frac{-\phi_3}{(1+\phi_3)}\frac{k_{95}}{(1+\nu)} & 0 & 0 & 0 & \frac{-\phi_3}{(1+\phi_3)}\frac{k_{99}}{(1+\nu)} & & & \\
0 & 0 & 0 & \frac{-k_{10,4}}{(1+\nu)} & 0 & 0 & 0 & 0 & 0 & \frac{-k_{10,10}}{(1+\nu)} & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\phi_3}{(1+\phi_3)}\frac{k_{11,9}}{(1+\nu)} & 0 & \frac{-3\phi_3 k_{11,11}}{(1+\phi_3)(4+\phi_3)(1+\nu)} & \\
0 & \frac{-\phi_2}{(1+\phi_2)}\frac{k_{12,2}}{(1+\nu)} & \frac{-\phi_3}{(1+\phi_3)}\frac{k_{11,3}}{(1+\nu)} & 0 & \frac{-3\phi_3 k_{11,5}}{(1+\phi_3)(2-\phi_3)(1+\nu)} & \frac{-3\phi_2 k_{12,6}}{(1+\phi_2)(2-\phi_2)(1+\nu)} & 0 & \frac{-\phi_2}{(1+\phi_2)}\frac{k_{12,8}}{(1+\nu)} & 0 & 0 & 0 & \frac{-3\phi_2 k_{12,12}}{(1+\phi_2)(4+\phi_2)(1+\nu)}
\end{array}\right]
$$

Table 4-6 Beam Element Sensitivity Matrix, $\dfrac{\partial\left[k_{beam}\right]}{\partial SF_2}$

$$
\begin{bmatrix}
0 & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & \\
0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{33}}{SF_2} & & & & & \text{Symmetric} & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{53}}{SF_2} & 0 & \frac{3\Phi_3}{(1+\Phi_3)(4+\Phi_3)}\frac{k_{55}}{SF_2} & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{93}}{SF_2} & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{95}}{SF_2} & 0 & 0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{99}}{SF_2} & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{11,3}}{SF_2} & 0 & \frac{3\Phi_3}{(1+\Phi_3)(2-\Phi_3)}\frac{k_{11,5}}{SF_2} & 0 & 0 & 0 & \frac{\Phi_3}{(1+\Phi_3)}\frac{k_{11,9}}{SF_2} & 0 & \frac{3\Phi_3}{(1+\Phi_3)(4+\Phi_3)}\frac{k_{11,11}}{SF_2} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Table 4-7 Beam Element Sensitivity Matrix, $\dfrac{\partial [k_{beam}]}{\partial SF_3}$

$$
\begin{bmatrix}
0 & & & & & & & & & & & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{22}}{SF_3} & & & & & & & & & & \\
0 & 0 & 0 & & & & & \text{Symmetric} & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{62}}{SF_3} & 0 & 0 & 0 & \frac{3\Phi_2}{(1+\Phi_2)(4+\Phi_2)}\frac{k_{66}}{SF_3} & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{82}}{SF_3} & 0 & 0 & 0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{86}}{SF_3} & 0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{88}}{SF_3} & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{12,2}}{SF_3} & 0 & 0 & 0 & \frac{3\Phi_2}{(1+\Phi_2)(2-\Phi_2)}\frac{k_{12,6}}{SF_3} & 0 & \frac{\Phi_2}{(1+\Phi_2)}\frac{k_{12,8}}{SF_3} & 0 & 0 & 0 & \frac{3\Phi_2}{(1+\Phi_2)(4+\Phi_2)}\frac{k_{12,12}}{SF_3}
\end{bmatrix}
$$

Table 4-8 Beam Element Sensitivity Matrix, $\dfrac{\partial\left[k_{beam}\right]}{\partial J}$

$$
\begin{bmatrix}
0 & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & \frac{k_{44}}{J} & & & & & \text{Symmetric} & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & \frac{k_{10,4}}{J} & 0 & 0 & 0 & 0 & 0 & \frac{k_{10,10}}{J} & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

4.1.3 Rotation of the Beam Property Sensitivity Matrices

Two types of reference frames are usually used in a structural dynamics problem. The first is the global or system reference frame which is assumed to be fixed in inertial space. The second type is the member reference frame. The orientation of the axes of this frame is defined with respect to the particular structural member (e.g. see Figure 4-1) and in our problem these axes, X_m, Y_m, and Z_m, correspond to the centroidal axis and the two cross-sectional principal axes of the member. It is convenient in matrix structure problems to first define all inertial and stiffness properties for a particular member in terms of the member reference frame (X_m, Y_m, Z_m) and then to systematically rotate the properties to corresponding expressions in the structure reference frame (X_s, Y_s, Z_s). Such a rotation procedure is followed in the development of the member property sensitivity matrices. First, as described in the previous section, the beam property sensitivity matrix is developed with respect to the member reference frame (X_m, Y_m, Z_m). In this section we discuss its rotation into the system, or global, reference frame.

Detailed derivation of equations for the rotation of force and displacement matrices can be found elsewhere (e.g. see Reference 4 , pages 18 - 31). Only the additional steps relevant to this task are presented in this section.

The rotation of the member property sensitivity matrix from a member to a global reference frame parallels the rotation of the corresponding stiffness matrix. First, the orientation of the member reference frame is established with respect to the system reference frame. This is accomplished by a sequence of three distinct rotations from one reference frame to the other contained in a rotation matrix, [R]. [R] has the form

$$[R] = \begin{bmatrix} [\gamma] & & & \text{Symmetric} \\ [0] & [\gamma] & & \\ [0] & [0] & [\gamma] & \\ [0] & [0] & [0] & [\gamma] \end{bmatrix} \qquad (4\text{-}1)$$

where $[\gamma]$ is a 3 x 3 matrix representing the coordinate transformation from the X_m, Y_m, Z_m frame to the X_s, Y_s, Z_s frame. Three distinct points in space are needed to define this rotation (Figure 4-2). Two of the points are on the beam centroidal axis, located at each end of the member. The third point must be defined away from the member axis and is used to orient the cross-section relative to the axis. This point normally lies in a plane formed by the beam axis and one of the principal axes of the beam cross-section.

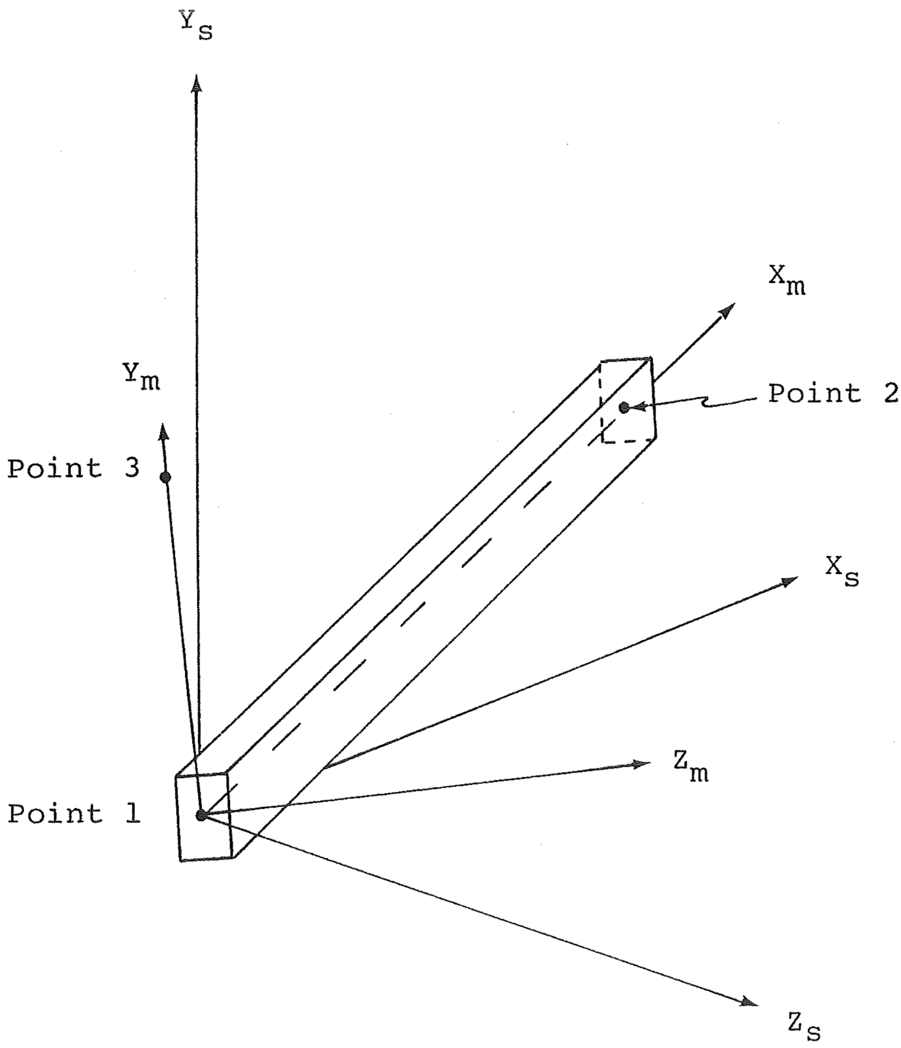


Figure 4-2 Three-Point Definition of Member Reference Frame

The coordinate numbering of the beam stiffness matrix (node 1: 3 transl., 3 rot.; node 2: 3 transl., 3 rot.) and the straight centroidal axis of the beam permit the repeated use of $[\gamma]$ consecutively down the diagonal of $[R]$.

The matrix equation defining the necessary rotation operations is

$$\left[\frac{\partial (k)_r}{\partial p_j}\right]_{beam} = [R]' \left[\frac{\partial (k)}{\partial p_j}\right]_{beam} [R] \qquad (4\text{-}2)$$

where $\left[\frac{\partial (k)}{\partial p_j}\right]_{beam} \equiv \frac{\partial}{\partial p_j} [K_{beam}]$

Once all of the elements of $\left[\frac{\partial (k)_r}{\partial p_j}\right]_{beam}$ have been computed, they are stored in a single column in a new matrix of the form shown in Equation (2-16).

The form of this matrix, for the beam, is shown in Equation (4-3).

$$\left[\frac{\partial (k)_r}{\partial (p)}\right]_{beam} = \begin{bmatrix} \frac{\partial k_{11_r}}{\partial E} & \frac{\partial k_{11_r}}{\partial A} & \frac{\partial k_{11_r}}{\partial I_2} & \frac{\partial k_{11_r}}{\partial I_3} & \frac{\partial k_{11_r}}{\partial \nu} & \cdots \\ \frac{\partial k_{12_r}}{\partial E} & \frac{\partial k_{12_r}}{\partial A} & \frac{\partial k_{12_r}}{\partial I_2} & \frac{\partial k_{12_r}}{\partial I_3} & \frac{\partial k_{12_r}}{\partial \nu} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \frac{\partial k_{nn_r}}{\partial E} & \frac{\partial k_{nn_r}}{\partial A} & \frac{\partial k_{nn_r}}{\partial I_2} & \frac{\partial k_{nn_r}}{\partial I_3} & \frac{\partial k_{nn_r}}{\partial \nu} & \cdots \end{bmatrix} \qquad (4\text{-}3)$$

Since $\left[\frac{\partial (k)_r}{\partial p_j}\right]_{beam}$ is symmetrical, only elements on the diagonal and on one side of the diagonal need to be stored in forming $\left[\frac{\partial (k)_r}{\partial (p)}\right]_{beam}$.

4.1.4 Beam Mass Property Sensitivity Matrix

The mass in this analysis is concentrated at the nodes. Hence the system mass matrix is diagonal and each element along the diagonal represents the mass or inertia corresponding to a dynamic degree of freedom. The masses and inertias are input in system coordinates and therefore the component mass matrix is not rotated as is the component stiffness matrix.

The mass at the node is treated as the random variable. Since the mass in the three translational directions is by definition the same, the transformation is unity. For the three rotational degrees of freedom the mass is a factor in the moment of inertia about each of the three axes, thus

$$\frac{\partial I}{\partial m} = \frac{I}{m}$$

Looking at the beam element we have two nodes with the mass distributed equally at both ends. The unconstrained mass matrix for structural element (i) in system coordinates is

$$\left[{}^{(i)}M\right] = \begin{bmatrix} m_{11} & & & & & & & \\ & m_{22} & & & & & 0 & \\ & & m_{33} & & & & & \\ & & & m_{44} & & & & \\ & & & & \ddots & & & \\ & 0 & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & m_{12,12} \end{bmatrix}$$

Elements m_{11}, m_{22}, m_{33}, m_{77}, m_{88}, and m_{99} are all equal and each equal half the total beam mass. Elements m_{44}, m_{55}, $m_{10,10}$, $m_{11,11}$ are equal and equivalent to some specified moment of inertia about the X_S, Y_S axes at each end of the beam. Elements m_{66} and $m_{12,12}$ are equal and equivalent to a specified moment of inertia about the Z_S axes at each end. These inertias generally contribute very little to the dynamic characteristics of the system and in many structural dynamic programs are set equal to zero in order to eliminate dynamic degrees of freedom and reduce the size of the model. In this model, however, no such reduction takes place and each of these inertias are considered.

The property sensitivity matrix for mass shows only one sensitivity and that is to the mass, at the node, thus

$$\left\{\frac{\partial(m)}{\partial m}\right\}_{beam} = \left\{\begin{array}{c} 1 \\ 1 \\ 1 \\ \dfrac{m_{44}}{m_{11}} \\ \dfrac{m_{55}}{m_{11}} \\ \dfrac{m_{66}}{m_{11}} \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \dfrac{m_{12,12}}{m_{11}} \end{array}\right\}$$

4.1.5 Synthesis of the Stiffness and Mass Property Sensitivity Matrices -- Beam

The two matrices $\left[\frac{\partial(k)_r}{\partial(p)}\right]_{beam}$ and $\left\{\frac{\partial(m)}{\partial m}\right\}_{beam}$ developed in Sections 4.13 and 4.14 are combined into a single matrix $\left[\frac{\partial(k_r,m)}{\partial(p)}\right]_{beam}$ as the next step in the development. Each column in this new matrix contains a set of partial derivatives, all with respect to the same independent variable. The first eight variables are the stiffness oriented variables E, I_2, I_3, etc. and the ninth is the mass.

The matrix is partitioned as shown below.

$$
\left[\frac{\partial (k_r, m)}{\partial (p)}\right]_{\text{beam}} =
\begin{array}{cc}
\begin{array}{cc} \longleftarrow 8 \text{ columns} \longrightarrow & \longleftarrow 1 \text{ column} \longrightarrow \end{array} & \\
\left[\begin{array}{c|c}
\left[\dfrac{\partial (k) r}{\partial (p)}\right] & \{0\} \\
\hline
[0] & \left\{\dfrac{\partial (m)}{\partial m}\right\}
\end{array}\right] & \begin{array}{l} \text{up to} \\ 78 \text{ rows} \\ \text{up to} \\ 12 \text{ rows} \end{array}
\end{array}
$$

The matrix will always have nine columns but the number of rows will depend upon the constraints upon the beam element. If there are no constraints the number of rows in each column of stiffness partials will be $1/2(n^2 + n) = 78$ where n, the number of unconstrained degrees of freedom in this case, is equal to 12. In addition there will be 12 mass partial derivatives (one for each unconstrained degree of freedom) making the total of the number of rows reach 90. This 90 x 9 matrix is equivalent to the matrix, $[C]$, in Equations (2-20) and (2-21). The number of columns must be conformable to the property covariance matrix to be developed in Section 5 and the number of rows must be conformable to the eigenvector and eigenvalue partial derivative matrix which will be discussed further in Section 6.

The order of the properties acting as independent variables in the partial derivatives in each of the columns, moving from left to right is: E, A, I_2, I_3, ν, SF_2, SF_3, J, and m.

4.2 Plate Element

4.2.1 Introduction

The plate element used in this program is a combination of a sandwich element developed by H.C. Martin in 1967 (Ref. 5) and a triangular element which resists in-plane forces (Ref. 3). The sandwich construction resists both bending and shear. This particular plate was selected because of its operational status in the STARDYNE structural dynamics program. (VIDAP was designed to be directly compatible with STARDYNE.)

The procedure for developing the plate property sensitivity matrix is similar to that for the beam except for the additional degrees of freedom and the form of the stiffness matrix elements. The steps are as follows: the stiffness matrix is defined as the sum of six matrices of geometric constants which are multiplied by constants containing the physical properties; the property sensitivity matrices are developed by differentiating the expression for the stiffness matrix; the differentiated matrix is pre and post multiplied by a rotation matrix to put the partial derivatives in system coordinates; and the resulting partial derivatives are stored in columns, each column representing a separate physical property.

The plate element used in this development is only one of many available today. It is, however, quite general and can be used in a variety of situations. If another plate element model is preferred, the method used for partial derivative development described on the following pages can be used with proper modifications and be entered into the VIDAP program as a general element as described in Section 4.3.

4.2.2 Plate Element Stiffness Matrix

A detailed physical and mathematical description of the Martin triangular plate element is presented in Ref. 5. In this section we shall discuss the physics of the element, its range of application, and mathematically define its stiffness matrix.

Figure 4-3 shows a schematic drawing of the sandwich plate element; it is composed of five basic structural components: two cover sheets and three shear webs. The coordinate identification is shown in Figure 4-4. Note the omission of the coordinate of rotation about the Z_m axis at each of the nodes. This means that each node has only five degrees of freedom and in the development of a stiffness matrix which allows six degrees of freedom at each node this will be equivalent to having zero plate stiffness in rotation about Z_m at each plate node.

The triangular top and bottom cover sheets, or flanges, are two dimensional plane stress finite elements. Each component sheet has two in-plane nodal degrees of freedom at its corners. The stiffness matrix corresponding to each cover sheet is given in Equation (4-4) (see Ref. 6, Turner, Clough, Martin, Topp).

Note that in all the succeeding matrices expressions such as x_{12} and y_{23} are obtained from the following identities

$$x_{\alpha\beta} = x_\alpha - x_\beta$$

$$y_{\alpha\beta} = y_\alpha - y_\beta$$

where x_α, x_β, y_α, y_β are coordinates of nodes α and β in the member coordinate system.

$$
\left[K_{cs}\right] = \frac{Eh\psi}{2}
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\
\left[\begin{array}{cccccc}
\lambda_1 x_{23}^2 + y_{23}^2 & & & & & \\
\lambda_2 x_{32} y_{23} & x_{23}^2 + \lambda_1 y_{23}^2 & & \text{Symmetric} & & \\
\lambda_1 x_{23} x_{31} + y_{23} y_{31} & \lambda_1 x_{13} y_{23} + \nu x_{23} y_{13} & \lambda_1 x_{13}^2 + y_{13}^2 & & & \\
\lambda_1 x_{23} y_{13} + \nu x_{13} y_{23} & x_{23} x_{31} + \lambda_1 y_{23} x_{31} & \lambda_2 x_{13} y_{31} & x_{13}^2 + \lambda_1 y_{13}^2 & & \\
\lambda_1 x_{12} x_{23} + y_{12} y_{23} & \lambda_1 x_{21} y_{23} + \nu x_{32} y_{12} & \lambda_1 x_{12} x_{31} + y_{12} y_{31} & \lambda_1 x_{12} y_{13} + \nu x_{13} y_{12} & \lambda_1 x_{12}^2 + y_{12}^2 & \\
\lambda_1 x_{32} y_{12} + \nu x_{21} x_{23} & x_{12} x_{23} + \lambda_1 y_{12} y_{23} & \lambda_1 x_{13} y_{12} + \nu x_{12} y_{13} & x_{12} x_{31} + \lambda_1 y_{12} y_{31} & \lambda_2 x_{21} y_{12} & x_{12}^2 + \lambda_1 y_{12}^2
\end{array}\right]
\end{array}
\quad (4\text{-}4
$$

where $x_{\alpha\beta} = x_\alpha - x_\beta$, etc. $\qquad \lambda_1 = \dfrac{1-\nu}{2}$

$\psi = \dfrac{1}{2(1-\nu^2)A} \qquad \lambda_2 = \dfrac{1+\nu}{2}$

A = area of triangle

ν = Poisson's ratio

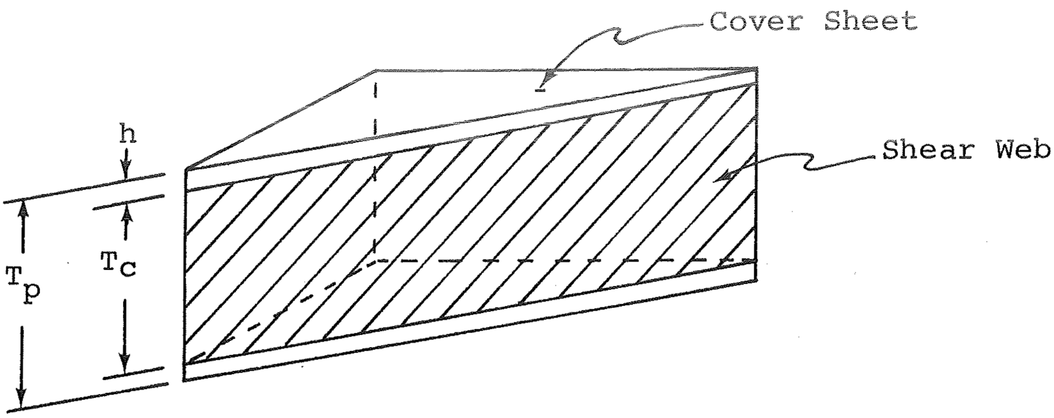


Figure 4-3 Schematic Drawing of a Martin Finite Element

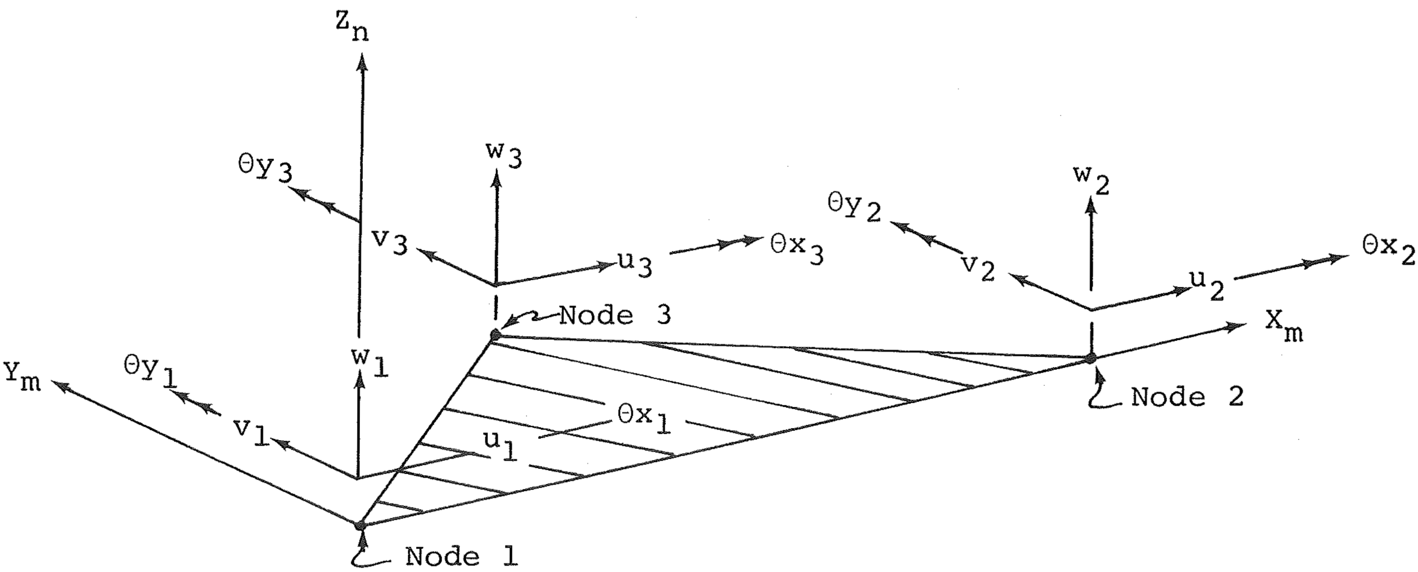


Figure 4-4 Coordinate Identification of the Triangular Plate Element

The cover sheet displacements and the sandwich element displacements (rotations) are related as shown in Figure 4-5

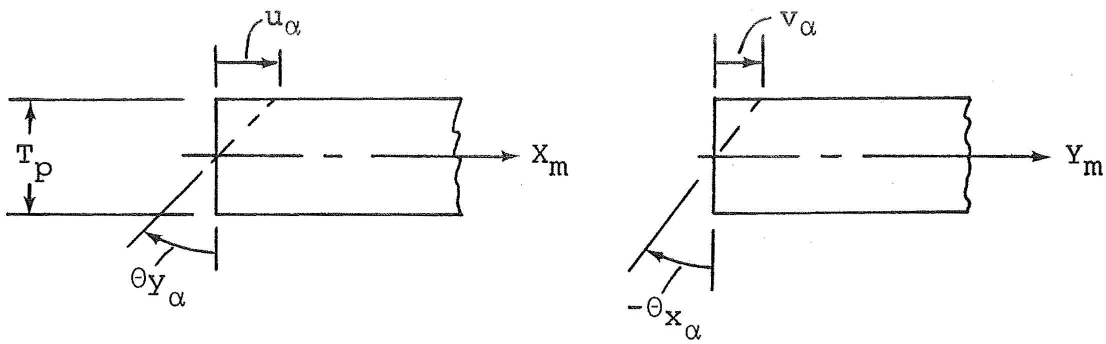


Figure 4-5 Relation Between Cover Sheet Displacements (u_α, v_α) and Corresponding Sandwich Element Displacements (θ_{x_α}, θ_{y_α})

These relations can be put in matrix form for the three nodes as follows

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \frac{Tp}{2} \left[\begin{array}{cc|cc|cc} 0 & 1 & & & & \\ -1 & 0 & & & & \\ \hline & & 0 & 1 & & \\ & & -1 & 0 & & \\ \hline & & & & 0 & 1 \\ & & & & -1 & 0 \end{array}\right] \begin{Bmatrix} \theta_{x_1} \\ \theta_{y_1} \\ \theta_{x_2} \\ \theta_{y_2} \\ \theta_{x_3} \\ \theta_{y_3} \end{Bmatrix} \qquad (4\text{-}5)$$

or $\{u\} = \frac{Tp}{2} [T]\{\theta\}$

The bending stiffness for the plate is constructed from [T] and [K_s]. The derivation is given in Ref. 5, pp 2-6. The stiffness in bending expanded to all fifteen degrees of freedom is shown in Equation (4-6) on the next page.

$$
\left[K_b\right] = \frac{ET_p^{\,3}\psi}{24}
\begin{array}{c}
\begin{array}{ccccccccccccccc}
u_1 & v_1 & w_1 & \theta_{x_1} & \theta_{y_1} & u_2 & v_2 & w_2 & \theta_{x_2} & \theta_{y_2} & u_3 & v_3 & w_3 & \theta_{x_3} & \theta_{y_3}
\end{array}\\
\left[\begin{array}{ccccccccccccccc}
0 & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & & & & \\
0 & 0 & 0 & x_{23}^2+\lambda_1 y_{23}^2 & & & & & & & & & & & \\
0 & 0 & 0 & -\lambda_2 x_{32} y_{23} & \lambda_1 x_{23}^2+y_{23}^2 & & & & & & & & \text{Symmetric} & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & x_{23}x_{31}+\lambda_1 y_{23}y_{31} & -\lambda_1 x_{23}y_{13}-\nu x_{13}y_{23} & 0 & 0 & 0 & x_{13}^2+\lambda_1 y_{13}^2 & & & & & & \\
0 & 0 & 0 & -\lambda_1 x_{13}y_{23}-\nu x_{23}y_{13} & \lambda_1 x_{23}x_{31}+y_{23}y_{31} & 0 & 0 & 0 & -\lambda_2 x_{13}y_{31} & \lambda_1 x_{13}^2+y_{13}^2 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & x_{12}x_{23}+\lambda_1 y_{12}y_{23} & -\lambda_1 x_{32}y_{12}-\nu x_{21}y_{23} & 0 & 0 & 0 & x_{12}x_{31}+\lambda_1 y_{12}y_{31} & -\lambda_1 x_{12}y_{12}-\nu x_{12}y_{13} & 0 & 0 & 0 & x_{12}^2+\lambda_1 y_{12}^2 & \\
0 & 0 & 0 & -\lambda_1 x_{21}y_{23}-\nu x_{32}y_{12} & \lambda_1 x_{12}x_{23}+y_{12}y_{23} & 0 & 0 & 0 & -\lambda_1 x_{12}y_{13}-\nu x_{13}y_{12} & \lambda_1 x_{12}x_{31}+y_{12}y_{31} & 0 & 0 & 0 & -\lambda_2 x_{21}y_{12} & \lambda_1 x_{12}^2+y_{12}^2
\end{array}\right]
\end{array}
\qquad (4\text{-}6)
$$

where $\psi = \dfrac{1}{2(1-\nu^2)A}$

$\lambda_1 = \dfrac{1-\nu}{2}$

$\lambda_2 = \dfrac{1+\nu}{2}$

A = area of the triangle

Note: The matrix expression above is identified later on in the text as $[K_2]$. That is,

$$[K_b] = \frac{ET_p^{\,3}\psi}{24}[K_2]$$

Three pure shear structural components are used for the web of the element. As the name implies, these components only resist shearing deformation. Each shear component connects two corners of the upper cover sheet with two corners of the lower cover sheet. The stiffness matrix of a typical shear component is given in Equation (4-7)

$\left(\frac{Aw}{L}\right)_{1-2}$ is obtained from the following expression

$$\begin{Bmatrix} \left(\frac{Aw}{L}\right)_{1-2} \\ \left(\frac{Aw}{L}\right)_{2-3} \\ \left(\frac{Aw}{L}\right)_{3-1} \end{Bmatrix} = \frac{T_p}{8A} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} L_{1-2}^{2} \\ L_{2-3}^{2} \\ L_{3-1}^{2} \end{Bmatrix} \qquad (4-8)$$

where A is the surface area of the plate and $L_{\alpha\beta}$ is the linear distance from node α to node β.

The stiffness matrices in Equations (4-6) and (4-7) are for out-of-plane loads only. The stiffness matrices due to in-plane loads are obtained from Reference 3, p. 86. The first, Equation (4-9) is the stiffness due to normal stresses and the second, Equation (4-10) is the stiffness due to shearing stresses.

$$
\left[Kw\right]_{1-2} = \frac{G}{4}\left(\frac{Aw}{L}\right)_{1-2}
\begin{array}{ccccccccccccccc}
u_1 & v_1 & w_1 & \Theta x_1 & \Theta y_1 & u_2 & v_2 & w_2 & \Theta x_2 & \Theta y_2 & u_3 & v_3 & w_3 & \Theta x_3 & \Theta y_3 \\
\end{array}
$$

$$
\begin{bmatrix}
0 & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & & & & \\
0 & 0 & 4 & & & & & & \text{Symmetric} & & & & & & \\
0 & 0 & 2y_{21} & y_{21}^2 & & & & & & & & & & & \\
0 & 0 & -2x_{21} & -x_{21}y_{21} & x_{21}^2 & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & -4 & -2y_{21} & 2x_{21} & 0 & 0 & 4 & & & & & & & \\
0 & 0 & 2y_{21} & y_{21}^2 & -x_{21}y_{21} & 0 & 0 & -2y_{21} & y_{21}^2 & & & & & & \\
0 & 0 & -2x_{21} & -x_{21}y_{21} & x_{21}^2 & 0 & 0 & 2x_{21} & -x_{21}y_{21} & x_{21}^2 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\quad (4\text{-}7)
$$

Where the subscript $(\ )_{1-2}$ signifies the effective shear web between nodes 1 and 2. See Equation (4-8) for further definition.

$$[K_n] = \frac{E(T_p-T_c)\psi}{2} \begin{array}{c} \begin{array}{ccccccccccccccc} u_1 & v_1 & w_1 & \Theta_{x_1} & \Theta_{y_1} & u_2 & v_2 & w_2 & \Theta_{x_2} & \Theta_{y_2} & u_3 & v_3 & w_3 & \Theta_{x_3} & \Theta_{y_3} \end{array} \\ \begin{bmatrix} y_{23}^2 & & & & & & & & & & & & & & \\ -\nu y_{23}x_{23} & x_{23}^2 & & & & & & & & & & & & & \\ 0 & 0 & 0 & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & & & & \text{Symmetric} & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\ y_{23}y_{31} & -\nu x_{23}y_{31} & 0 & 0 & 0 & y_{31}^2 & & & & & & & & & \\ -\nu y_{23}x_{31} & x_{23}x_{31} & 0 & 0 & 0 & -\nu y_{31}x_{31} & x_{31}^2 & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ -y_{23}y_{21} & \nu x_{23}y_{21} & 0 & 0 & 0 & -y_{21}y_{31} & \nu y_{21}x_{31} & 0 & 0 & 0 & y_{21}^2 & & & & \\ \nu y_{23}x_{21} & -x_{23}x_{21} & 0 & 0 & 0 & \nu x_{21}y_{31} & -x_{21}x_{31} & 0 & 0 & 0 & -\nu y_{21}x_{21} & x_{21}^2 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (4\text{-}9)$$

where $\psi = \dfrac{1}{2(1-\nu)A}$

Note: The matrix expression above is identified later on in the text as $[K_3]$. That is,

$$[K_n] = \frac{E(T_p-T_c)\psi}{2}\,[K_3]$$

$$
[K_s] = \frac{E(T_p-T_c)(1-\nu)\psi}{4}
\begin{array}{c}
\begin{array}{ccccccccccccccc}
u_1 & v_1 & w_1 & \theta_{x_1} & \theta_{y_1} & u_2 & v_2 & w_2 & \theta_{x_1} & \theta_{y_1} & u_3 & v_3 & w_3 & \theta_{x_1} & \theta_{y_1}
\end{array} \\
\left[\begin{array}{ccccccccccccccc}
x_{23}^2 & & & & & & & & & & & & & & \\
-x_{23}y_{23} & y_{23}^2 & & & & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & \text{Symmetric} & & & & & \\
x_{23}x_{31} & -y_{23}y_{31} & 0 & 0 & 0 & x_{31}^2 & & & & & & & & & \\
-x_{23}y_{31} & y_{23}y_{31} & 0 & 0 & 0 & -x_{31}y_{31} & y_{31}^2 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
-x_{23}x_{21} & y_{23}x_{21} & 0 & 0 & 0 & -x_{21}x_{31} & x_{21}y_{31} & 0 & 0 & 0 & x_{21}^2 & & & & \\
x_{23}y_{21} & -y_{23}x_{21} & 0 & 0 & 0 & y_{21}x_{31} & -x_{21}y_{31} & 0 & 0 & 0 & -x_{21}y_{21} & y_{21}^2 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\qquad (4\text{-}10)
$$

where $\psi = \dfrac{1}{2(1-\nu^2)A}$

Note: The matrix expression above is identified later on in the text as $[K_4]$.

That is $[K_s] = \dfrac{E(T_p-T_c)(1-\nu)\psi}{4} [K_4]$

The complete stiffness matrix is constructed from the bending, the normal, the in-plane shear, and the out-of-plane shear stiffness matrices. The summation is shown in Equation (4-11).

$$[K_{plate}] = [K_b] + [K_n] + [K_s] + [K_w]_{1-2} + [K_w]_{2-3} + [K_w]_{3-1} \tag{4-11}$$

Equation (4-11) produces a 15 x 15 stiffness matrix in the member reference frame, where the origin is at node 1, the line between nodes 1 and 2 forms the X_m axis and the plate lies in the X_mY_m plane.

4.2.3 Plate Stiffness Property Sensitivity Matrix

Five properties of the plate stiffness may be considered as random variables

- G - modulus of rigidity of the core material
- T_c - thickness of the core
- T_p - total thickness of the plate
- E - modulus of elasticity of the face sheets
- ν - Poisson's ratio

As in the case of the beam, nodal dimensions are not considered random.

The derivation of the plate element stiffness sensitivity matrix follows the same principle as that used for the beam but the procedure is slightly different. A quick glance at Equations (4-6, 7, 9 and 10) will reveal that most of the time the properties listed above are found in the constants multiplying the entire matrix rather than within the matrix elements themselves. Only Poisson's ratio lies within these constituent matrices.

Let

$$[K_w] = [K_w]_{1-2} + [K_w]_{2-3} + [K_w]_{3-1} = \frac{GTp}{32A}[K_1] \tag{4-12}$$

$$[K_b] = \frac{ETp^3\psi}{24}[K_2] \tag{4-13}$$

$$[K_n] = \frac{E(T_p - T_c)\psi}{2} [K_3] \tag{4-14}$$

$$[K_3] = \frac{E(T_p - T_c)(1-\nu)\psi}{4} [K_4] \tag{4-15}$$

$[K_4]$ contains only geometric terms and is the matrix expression in Equation (4-10). $[K_1]$ is a composite of the matrix expression for the three shear webs and is shown in Equation (4-16). $[K_1]$ contains only geometric terms too and thus will not change in form when $[K_{plate}]$ is differentiated. The matrices $[K_2]$ and $[K_3]$ both contain the single physical constant ν and thus will change form only when differentiated with respect to ν. The matrices $\frac{\partial [K_2]}{\partial \nu}$ and $\frac{\partial [K_3]}{\partial \nu}$ are shown in Equations (4-17) and (4-18).

$$
[K_1] = \begin{array}{c}
\begin{array}{ccccccccccccccc}
u_1 & v_1 & w_1 & \theta_{x_1} & \theta_{y_1} & u_2 & v_2 & w_2 & \theta_{x_2} & \theta_{y_2} & u_3 & v_3 & w_3 & \theta_{x_3} & \theta_{y_3}
\end{array} \\
\left[\begin{array}{ccccccccccccccc}
0 & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & & & & \\
0 & 0 & \frac{4}{L'_{12}} + \frac{4}{L'_{31}} & & & & & & & & & & & & \\
0 & 0 & \frac{2y_{21}}{L'_{12}} - \frac{2y_{13}}{L'_{31}} & \frac{y_{21}^2}{L'_{12}} + \frac{y_{13}^2}{L'_{31}} & & & & & & & & & & & \\
0 & 0 & -\frac{2x_{21}}{L'_{12}} + \frac{2x_{13}}{L'_{31}} & -\frac{x_{21}y_{21}}{L'_{12}} - \frac{x_{13}y_{13}}{L'_{31}} & \frac{x_{21}^2}{L'_{12}} + \frac{x_{13}^2}{L'_{31}} & & & & & \text{Symmetric} & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & -\frac{4}{L'_{12}} & -\frac{2y_{21}}{L'_{12}} & \frac{2x_{21}}{L'_{12}} & 0 & 0 & \frac{4}{L'_{23}} + \frac{4}{L'_{12}} & & & & & & & \\
0 & 0 & \frac{2y_{21}}{L'_{12}} & \frac{y_{21}^2}{L'_{12}} & -\frac{x_{21}y_{21}}{L'_{12}} & 0 & 0 & -\frac{2y_{21}}{L'_{12}} + \frac{2y_{32}}{L'_{23}} & \frac{y_{21}^2}{L'_{12}} + \frac{y_{32}^2}{L'_{23}} & & & & & & \\
0 & 0 & -\frac{2x_{21}}{L'_{12}} & -\frac{x_{21}y_{21}}{L'_{12}} & \frac{x_{21}^2}{L'_{12}} & 0 & 0 & \frac{2x_{21}}{L'_{12}} - \frac{2x_{32}}{L'_{23}} & -\frac{x_{21}y_{21}}{L'_{12}} - \frac{x_{32}y_{32}}{L'_{23}} & \frac{x_{32}^2}{L'_{23}} + \frac{x_{21}^2}{L'_{12}} & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & -\frac{4}{L'_{31}} & \frac{2y_{13}}{L'_{31}} & -\frac{2x_{13}}{L'_{31}} & 0 & 0 & -\frac{4}{L'_{23}} & -\frac{2y_{32}}{L'_{23}} & \frac{2x_{32}}{L'_{23}} & 0 & 0 & \frac{4}{L'_{23}} + \frac{4}{L'_{31}} & & \\
0 & 0 & -\frac{2y_{13}}{L'_{31}} & \frac{y_{13}^2}{L'_{31}} & -\frac{x_{13}y_{13}}{L'_{31}} & 0 & 0 & \frac{2y_{32}}{L'_{23}} & \frac{y_{32}^2}{L'_{23}} & -\frac{x_{32}y_{32}}{L'_{23}} & 0 & 0 & -\frac{2y_{32}}{L'_{23}} + \frac{2y_{13}}{L'_{31}} & \frac{y_{13}^2}{L'_{31}} + \frac{y_{32}^2}{L'_{23}} & \\
0 & 0 & \frac{2x_{13}}{L'_{31}} & -\frac{x_{13}y_{13}}{L'_{31}} & \frac{x_{13}^2}{L'_{31}} & 0 & 0 & -\frac{2x_{32}}{L'_{23}} & -\frac{x_{32}y_{32}}{L'_{23}} & \frac{x_{32}^2}{L'_{23}} & 0 & 0 & \frac{2x_{32}}{L'_{23}} - \frac{2x_{13}}{L'_{31}} & -\frac{x_{13}y_{13}}{L'_{31}} - \frac{x_{32}y_{32}}{L'_{23}} & \frac{x_{13}^2}{L'_{31}} + \frac{x_{32}^2}{L'_{23}}
\end{array}\right]
\end{array}
\qquad (4\text{-}16)
$$

where $\frac{1}{L'_{12}} = -L_{1-2}^2 + L_{2-3}^2 + L_{3-1}^2$, $\frac{1}{L'_{23}} = L_{1-2}^2 - L_{2-3}^2 + L_{3-1}^2$, $\frac{1}{L'_{31}} = L_{1-2}^2 + L_{2-3}^2 - L_{3-1}^2$

$x_{\alpha\beta} = x_\alpha - x_\beta$, $y_{\alpha\beta} = y_\alpha - y_\beta$

and $L_{\alpha\beta} = \sqrt{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}$

$$\frac{\partial[K_2]}{\partial\nu} = \begin{array}{c} \begin{array}{ccccccccccccccc} u_1 & v_1 & w_1 & \Theta_{x_1} & \Theta_{y_1} & u_2 & v_2 & w_2 & \Theta_{x_2} & \Theta_{y_2} & u_3 & v_3 & w_3 & \Theta_{x_3} & \Theta_{y_3} \end{array} \\ \left[\begin{array}{ccccccccccccccc}
0 & & & & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & & & & \\
0 & 0 & 0 & -\frac{y_{23}^2}{2} & & & & & & & & & & & \\
0 & 0 & 0 & -\frac{x_{32}y_{23}}{2} & -\frac{x_{23}^2}{2} & & & & & \text{Symmetric} & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & -\frac{1}{2}y_{23}y_{31} & \frac{1}{2}x_{23}y_{13} - x_{13}y_{23} & 0 & 0 & 0 & -\frac{1}{2}y_{13}^2 & & & & & & \\
0 & 0 & 0 & \frac{1}{2}x_{13}y_{23} - x_{23}y_{13} & -\frac{1}{2}x_{23}x_{31} & 0 & 0 & 0 & -\frac{1}{2}x_{12}y_{31} & -\frac{1}{2}x_{13}^2 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & -\frac{1}{2}y_{12}y_{23} & \frac{1}{2}x_{32}y_{12} - x_{21}y_{23} & 0 & 0 & 0 & -\frac{1}{2}y_{12}y_{31} & \frac{1}{2}x_{13}y_{12} - x_{12}y_{13} & 0 & 0 & 0 & -\frac{1}{2}y_{12}^2 & \\
0 & 0 & 0 & \frac{1}{2}x_{21}y_{23} - x_{32}y_{12} & -\frac{1}{2}x_{12}x_{23} & 0 & 0 & 0 & \frac{1}{2}x_{12}y_{13} - x_{13}y_{12} & -\frac{1}{2}x_{12}x_{31} & 0 & 0 & 0 & -\frac{1}{2}x_{21}y_{12} & -\frac{1}{2}x_{12}^2
\end{array}\right] \end{array} \qquad (4\text{-}17)$$

$$
\frac{\partial [K_3]}{\partial \nu} =
\begin{array}{c}
\begin{array}{ccccccccccccccc}
u_1 & v_1 & w_1 & \Theta_{x_1} & \Theta_{y_1} & u_2 & v_2 & w_2 & \Theta_{x_2} & \Theta_{y_2} & u_3 & v_3 & w_3 & \Theta_{x_3} & \Theta_{y_3}
\end{array} \\
\left[\begin{array}{ccccccccccccccc}
0 & & & & & & & & & & & & & & \\
-y_{23}x_{23} & 0 & & & & & & & \text{Symmetric} & & & & & & \\
0 & 0 & 0 & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & & & & \\
0 & -x_{23}y_{31} & 0 & 0 & 0 & 0 & & & & & & & & & \\
-y_{23}x_{31} & 0 & 0 & 0 & 0 & -y_{31}x_{31} & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
0 & x_{23}y_{21} & 0 & 0 & 0 & 0 & y_{21}x_{31} & 0 & 0 & 0 & 0 & & & & \\
y_{23}x_{21} & 0 & 0 & 0 & 0 & x_{21}y_{31} & 0 & 0 & 0 & 0 & -y_{21}x_{21} & 0 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\qquad (4\text{-}18)
$$

The stiffness matrix can now be written in the following form

$$[K_{plate}] = \frac{GT_p}{32A}[K_1] + \frac{ET_p{}^3\psi}{24}[K_2] + \frac{E(T_p - T_c)\psi}{2}[K_3]$$

$$+ \frac{E(T_p - T_c)(1-\nu)\psi}{4}[K_4] \qquad (4\text{-}19)$$

The differentiation of (4-19) with respect to each of the five physical variables leads to the following five expressions

$$\frac{\partial[K_{plate}]}{\partial G} = \frac{T_p}{32A}[K_1] \qquad (4\text{-}20)$$

$$\frac{\partial[K_{plate}]}{\partial T_c} = -\frac{E\psi}{2}[K_3] - \frac{E(1-\nu)\psi}{4}[K_4] \qquad (4\text{-}21)$$

$$\frac{\partial[K_{plate}]}{\partial T_p} = \frac{G}{32A}[K_1] + \frac{ET_p{}^2\psi}{8}[K_2] + \frac{E\psi}{2}[K_3]$$

$$+ \frac{E(1-\nu)\psi}{4}[K_4] \qquad (4\text{-}22)$$

$$\frac{\partial[K_{plate}]}{\partial E} = \frac{T_p{}^3\psi}{24}[K_2] + \frac{(T_p - T_c)\psi}{2}[K_3]$$

$$+ \frac{(T_p - T_c)(1-\nu)\psi}{4}[K_4] \qquad (4\text{-}23)$$

$$\left[\frac{\partial(k)_{plate}}{\partial\nu}\right] \equiv \frac{\partial[K_{plate}]}{\partial\nu} = \frac{ET_p{}^3A\nu\psi^2}{6}[K_2] + \frac{ET_p{}^3\psi}{24}\frac{\partial[K_2]}{\partial\nu}$$

$$+ 2E(T_p-T_c)A\nu\psi^2[K_3] + \frac{E(T_p-T_c)\psi}{2}\frac{\partial[K_3]}{\partial\nu}$$

$$+ \frac{E(T_p - T_c)\psi}{4}\{8A\nu\psi(1-\nu) - 1\}[K_4]$$

$$(4\text{-}24)$$

4.2.4 Rotation of the Plate Property Sensitivity Matrices

The 3 x 3 matrix, $[\gamma]$, describing the transformation from the X_m, Y_m, Z_m frame to the X_s, Y_s, Z_s frame is developed in the same manner as that used for the beam with the three node points of the plate corresponding to the 3 reference points for the beam (see Figure 4-2). However, with the plate element there is no rotation about the Z_m axis even though the system permits such a rotation. Thus in transforming from member to system coordinates the stiffness matrix enlarges from 15 x 15 to 18 x 18. The rotation matrix, used to transform X_m, Y_m, Z_m, Θ_{x_m}, and Θ_{y_m} into X_s, Y_s, Z_s, Θ_{x_s}, Θ_{y_s}, and Θ_{z_s} is therefore

$$[\gamma_{plate}] = \begin{bmatrix} \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} & [0] \\ [0] & \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \end{bmatrix} \end{bmatrix} \qquad (4-25)$$

where the 3 x 3 $[\gamma]$ is identical to that developed for the beam.

The complete rotation matrix for the plate is

$$[R_{plate}] = \begin{bmatrix} [\gamma_{plate}] & [0] & [0] \\ [0] & [\gamma_{plate}] & [0] \\ [0] & [0] & [\gamma_{plate}] \end{bmatrix} \qquad (4-26)$$

$[R_{plate}]$ is a 15 x 18 matrix. The resulting stiffness matrix in system coordinates is

$$[K_{plate}\ (syst.)] = [R_{plate}]'[K_{plate}][R_{plate}] \qquad (4-27)$$

As in the case of the beam, the rotation procedure applied to the plate stiffness matrices carries over in the transformation of the partial derivatives into the system coordinates.

The equation is

$$\left[\frac{\partial (k)_r}{\partial p_i}\right]_{plate} = \left[R_{plate}\right]' \left[\frac{\partial (k)_{plate}}{\partial p_i}\right] \left[R_{plate}\right] \qquad (4\text{-}28)$$

Once all of the elements of $\left[\frac{\partial (k)_r}{\partial p_i}\right]_{plate}$ have been completed, they are stored in a single column in a new matrix of the form shown in Equation (2-16) and (4-3). Since $\left[\frac{\partial (k)_r}{\partial p_i}\right]_{plate}$ is symmetrical only the elements on the diagonal and on one side of the diagonal are stored in the formation of $\left[\frac{\partial (k)_r}{\partial (p)}\right]_{plate}$.

4.2.5 Plate Mass Property Sensitivity Matrix

As in the case of the beam, the mass of the plate is concentrated at the nodes. The plate element mass matrix is set up in system coordinates rather than the member reference frame to permit the utilization of a diagonal mass matrix. The unconstrained mass matrix is 18 x 18 to accomodate six dynamic degrees of freedom at each node.

The mass at the node is treated as the random variable and the partial derivatives of the mass elements are all taken with respect to the actual mass at the node. The property sensitivity matrix is a column with up to 18 elements as shown. It is formed in the same way as the beam except for three nodes rather than two. It is assumed that the mass of the plate is equally divided between the three nodes.

(over)

$$\left\{\frac{\partial (m)}{\partial m}\right\}_{plate} = \left\{\begin{array}{c} 1 \\ 1 \\ 1 \\ \dfrac{m_{44}}{m_{11}} \\ \cdot \\ \cdot \\ 1 \\ 1 \\ 1 \\ \dfrac{m_{10,10}}{m_{11}} \\ \cdot \\ \cdot \\ 1 \\ 1 \\ 1 \\ \dfrac{m_{16,16}}{m_{11}} \\ \cdot \\ \dfrac{m_{18,18}}{m_{11}} \end{array}\right\}$$

4.2.6 Synthesis of the Stiffness and Mass Property Sensitivity Matrices -- Plate

Section 4.1.5 describes the synthesis of the mass and stiffness property sensitivity matrices for the beam. The procedure for the plate is identical except for the dimensioning. The eighteen possible degrees of freedom of a plate requires an increase in the number of rows required to handle the partial derivatives for all the mass and stiffness elements. Thus the total number of rows can increase to 154 for the stiffness elements and 18 for the mass elements.

For convenience in the development of the VIDAP program it was decided to maintain the property covariance matrix as a 9 x 9 matrix. Therefore in the development of the plate property sensitivity matrix where there are only five properties which can affect stiffness and one which can affect mass, three columns will contain all zeros. The arrangement of the columns is shown below.

$$
\left[\frac{\partial(k_r, m)}{\partial(p)}\right]_{plate} =
\begin{array}{c}
\begin{array}{ccc} \overset{5}{\longleftarrow \text{columns} \longrightarrow} & \overset{3}{\longleftarrow \text{columns} \longrightarrow} & \overset{1}{\longleftarrow \text{column} \longrightarrow} \end{array} \\
\left[\begin{array}{c|c|c}
\left[\frac{\partial(k)r}{\partial(p)}\right] & [0] & \{0\} \\
\hline
[0] & [0] & \left\{\frac{\partial(m)}{\partial m}\right\}
\end{array}\right]
\end{array}
\begin{array}{l} \text{up to } 171 \text{ rows} \\ \text{up to } 18 \text{ rows} \end{array}
$$

The order of physical properties acting as independent variables in each of the first five columns moving from left to right is G, T_c, T_p, E, and ν. The independent variable in the ninth column is m.

4.3 General Stiffness Matrix

4.3.1 General Discussion

A particular structure may, or may not, be composed of only beam and plate elements. For this reason we shall outline the procedure required to formulate the sensitivity matrix of a general structure.

In formulating the stiffness matrix of any structure we visualize a sequence of unit generalized displacements being applied successively at generalized coordinates creating corresponding generalized forces (stiffness coefficients). Therefore, the different component parts of the structure can be imagined to overlay in the stiffness matrix. In order to demonstrate this and clarify further discussion, consider the example structure shown in Fig. 4-6. The stiffness matrix for the structure (neglecting shear deformations) is

$$\left[K_1\right] = \begin{bmatrix} \frac{4E_AI_A}{L_A} & & \text{Symmetric} & \\ -\frac{6E_AI_A}{L_A^2} & \frac{12E_AI_A}{L_A^3} + \frac{12E_BI_B}{L_B^3} & & \\ \frac{2E_AI_A}{L_A} & -\frac{6E_AI_A}{L_A^2} + \frac{6E_BI_B}{L_B^2} & \frac{4E_AI_A}{L_A} + \frac{4E_BI_B}{L_B} & \\ 0 & \frac{6E_BI_B}{L_B^2} & \frac{2E_BI_B}{L_B} & \frac{4E_BI_B}{L_B} \end{bmatrix}$$

Therefore, any statistical uncertainty in member A would affect rows and columns 1 through 3 while any uncertainty in member B affects rows and columns 2 through 4. If, for example, only the modulus of elasticity of member A was random the structure sensitivity matrix would be zero everywhere except rows and columns 1 through 3, i.e.

$$\frac{\partial\,[K]}{\partial E_A} = \frac{I_A}{L_A^3}\begin{bmatrix} 4L_A^2 & & \text{Symmetric} & \\ -6L_A & 12 & & \\ 2L_A^2 & -6L_A & 4L_A^2 & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

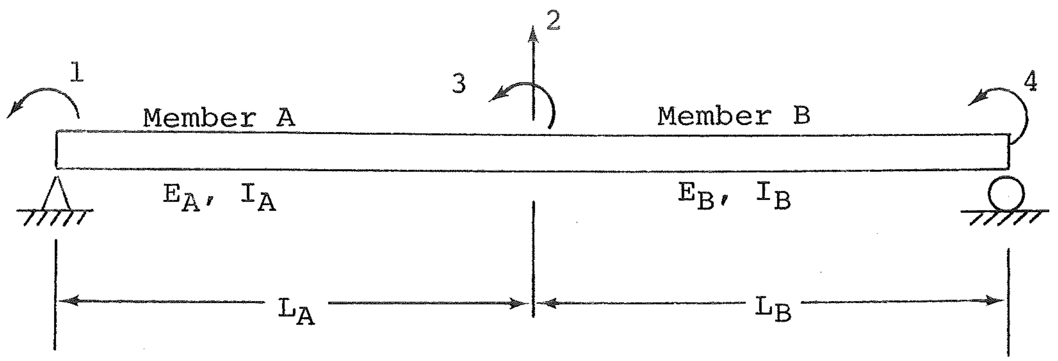


Figure 4-6 Example Structure

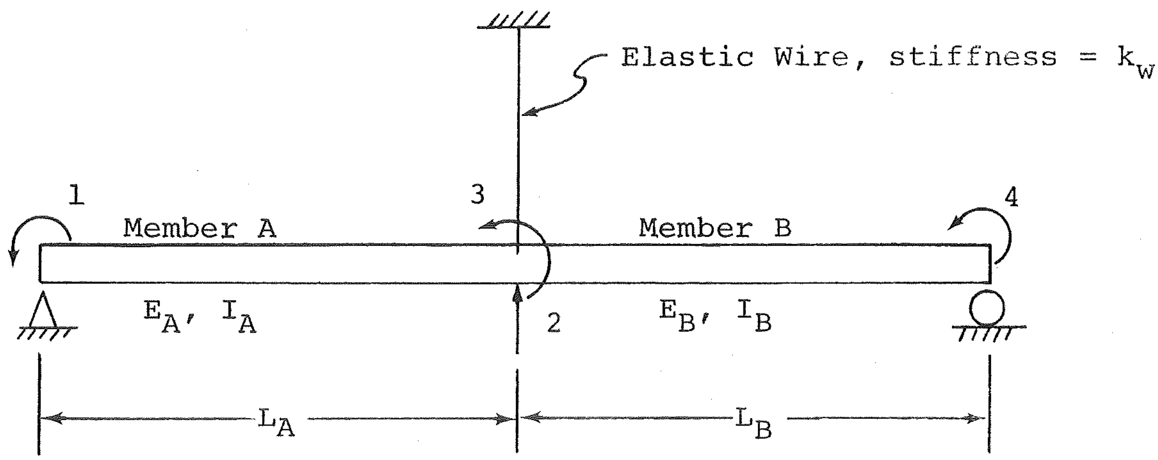


Figure 4-7 Example Structure with
Additional Stiffness Element

We can conclude using this example that only the rows and columns of the stiffness matrix associated with the generalized displacements which define the shape characteristics of the member whose parameter(s) are random are nonzero. By altering this physical system only slightly we can demonstrate the procedure used for a non-beam or plate random member.

Consider the structure shown in Figure 4-7. A vertical wire supplies additional stiffness to the beam at the midpoint. The structure (beam) stiffness matrix will now take the form

$$[K_2] = [K_1] + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_w & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Inspection of this stiffness matrix reveals that uncertainty in the wire properties will only affect element (2,2). Therefore,

$$\frac{\partial [K_s]}{\partial p} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\partial k_w}{\partial p} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad p = \text{random wire property}$$

This example demonstrates that if a general structural component which contributes stiffness to the structure has a random property, then only the row(s) and column(s) associated with generalized coordinates which prescribe its deformed shape are nonzero. And, a partial derivative of the elements in these rows and columns with respect to the random parameter result in the stiffness property sensitivity matrix.

4.3.2 Implementation

The VIDAP program is designed to handle the statistical characteristics of beam and plate elements and arbitrary stiffness matrices for general elements. If the option is to use an arbitrary stiffness matrix with some specified randomness, the property sensitivity matrix and the property covariance matrix must be developed outside the program, be multiplied together, and be entered as a single matrix into the program. This procedure means that the product $[C][^{(i)}\Sigma_p][C]'$ in Equations (2-20) and (2-21) is entered as a single matrix. An example of the development of this product is shown in the sample problem in Section 7.

If an arbitrary stiffness matrix is used, the compatibility of the elements in the product $[C][^{(i)}\Sigma_p][C]'$ with the partial derivatives of the eigenvalues and eigenvectors must be assured by hand preparation of the $[KR]$ and $[KS]$ matrices. These matrices are explained in Section 6 and are developed in the sample problem in Section 7.

5.0 PROPERTY COVARIANCE MATRICES

5.1 Introduction

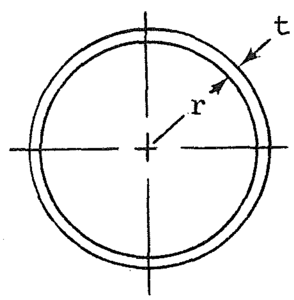
In Section 4 methods were presented which developed the partial derivatives relating mass and stiffness matrix elements to specific beam or plate properties. This completed the partial derivative development necessary for relating physical properties to modal properties by means of Equations (2-17) and (2-18) in Section 2. The remaining step therefore is the construction of the covariance matrix for the physical properties to be incorporated into Equations (2-21) and (2-22). The general form for the property covariance matrix is shown below.

$$[\Sigma_P] = \begin{bmatrix} \sigma_{p_1}{}^2 & \rho_{12}\sigma_{p_1}\sigma_{p_2} & \rho_{13}\sigma_{p_1}\sigma_{p_3} & \cdot \\ \rho_{12}\sigma_{p_1}\sigma_{p_2} & \sigma_{p_2}{}^2 & \rho_{23}\sigma_{p_2}\sigma_{p_3} & \cdot \\ \rho_{13}\sigma_{p_1}\sigma_{p_3} & \rho_{23}\sigma_{p_2}\sigma_{p_3} & \sigma_{p}{}^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

If properties such as E, ν, I, t, A, etc. are statistically independent, the correlation coefficients, ρ_{ij}, vanish and the matrix is diagonal. However, in most cases correlation does exist because, for example, A, I_2, I_3, and J depend upon t. This covariance varies with the cross-sectional configuration of the element and since the variety of configurations is limitless this part of the statistical operation is computed outside the VIDAP program and provided as an input in matrix form. The remainder of this section is devoted to tutoring the user in the development of these covariance matrices.

5.2 Development of a Property Covariance Matrix for a Tubular Beam Element

The physical properties used to describe the stiffness matrix for a beam are E, A, I_2, I_3, ν, SF_2, SF_3, and J. In the case of a tube the properties A, I_2, I_3, and J are all dependent upon the thickness, t, and the inside radius, r, of the tube.



For $t \ll r$

$$J = 2\pi r^2 t$$

$$I_2 = I_3 = \pi t r^3$$

$$A = 2\pi r t$$

$m = \rho 2\pi r t\, L \cdot 1/2$ (beam mass is split between the two nodes, this m is for a single node)

$$\frac{\partial J}{\partial r} = 4\pi r t$$

$$\frac{\partial J}{\partial t} = 2\pi r^2$$

$$\frac{\partial I_2}{\partial r} = 3\pi t r^2$$

$$\frac{\partial I_2}{\partial t} = \pi r^3$$

$$\frac{\partial A}{\partial r} = 2\pi t$$

$$\frac{\partial A}{\partial t} = 2\pi r$$

$$\frac{\partial m}{\partial r} = \pi \rho t L$$

$$\frac{\partial m}{\partial t} = \pi \rho r L$$

Assume E and ν are statistically independent of each other and all other variables. The form factors, SF_2 and SF_3, are functions of the geometric character and not of the specific measurements and in the case of a thin-walled tube are quite well known (no uncertainty). Therefore t, r, E, ν, and ρ are the independent random properties and can be described statistically in terms of a diagonal covariance matrix.

$$\left[\sum t, r, E, \nu, \rho\right] = \begin{bmatrix} \sigma_t^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_r^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_E^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_\nu^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\rho^2 \end{bmatrix}$$

Next describe the beam properties as functions of these independent properties.

$$\begin{Bmatrix} dE \\ dA \\ dI_2 \\ dI_3 \\ d\nu \\ dSF_2 \\ dSF_3 \\ dJ \\ dm \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 2\pi r & 2\pi t & 0 & 0 & 0 \\ \pi r^3 & 3\pi tr^2 & 0 & 0 & 0 \\ \pi r^3 & 3\pi tr^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2\pi r^2 & 4\pi rt & 0 & 0 & 0 \\ \pi\rho rL & \pi\rho tL & 0 & 0 & \pi rtL \end{bmatrix} \begin{Bmatrix} dt \\ dr \\ dE \\ d\nu \\ d\rho \end{Bmatrix}$$

$$= \left[\frac{\partial(E, A, I_2, \ldots)}{\partial(t, r, E, \ldots)}\right] \begin{Bmatrix} dt \\ dr \\ dE \\ d\nu \\ d\rho \end{Bmatrix}$$

Thus

$$[\Sigma p] = \left[\frac{\partial(E, A, I_2, \ldots)}{\partial(t, r, E, \ldots)}\right] [\Sigma t, r, G, \nu, \rho] \left[\frac{\partial(E, A, I_2, \ldots)}{\partial(t, r, E, \ldots)}\right]'$$

Note that $[\Sigma p]$ is always a 9 x 9 matrix and the entire preparation of $[\Sigma p]$ must be made prior to input into the VIDAP program.

5.3 Development of a Property Covariance Matrix for a Sandwich Plate Element

The physical properties used to describe the stiffness matrix for a sandwich plate are G(core), T_c (thickness of the core), T_p (total thickness), E (face sheets), and ν (face sheets). The probable random variable will be face sheet thickness (h), core thickness, G, E, and ν. Building a covariance matrices out of these latter variables we have

$$\left[\sum G, T_c, h, E, \nu, m\right] = \begin{bmatrix} \sigma_G^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{T_c}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_h^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_E^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\nu^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_m^2 \end{bmatrix}$$

The plate properties can be written as functions of these independent properties as follows

$$\begin{Bmatrix} dG \\ dT_c \\ dT_p \\ dE \\ d\nu \\ 0 \\ 0 \\ 0 \\ dm \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} dG \\ dT_c \\ dh \\ dE \\ d\nu \\ dm \end{Bmatrix}$$

$$= \left[\frac{\partial(G, T_c, T_p, E, \nu, m)}{\partial(G, T_c, h, E, \nu, m)}\right] \begin{Bmatrix} dG \\ dT_c \\ dh \\ dE \\ d\nu \\ dm \end{Bmatrix}$$

Recall that the property covariance matrix is always 9 x 9 and therefore the properties affecting the plate stiffness are in the first five rows in the above matrix, whereas the mass is in the ninth row. The resulting covariance matrix will have all zeros in the 6th, 7th, and 8th rows and columns but this will be conformable with $\left[\frac{\partial (k_r, m)}{\partial (p)}\right]_{plate}$ as developed in Section 4.2.6.

Thus

$$\left[\Sigma_p\right] = \left[\frac{\partial (G,\ T_c,\ T_p,\ \cdot\ \cdot)}{\partial (G,\ T_c,\ h,\ \cdot\ \cdot)}\right]\left[\Sigma_{G,\ T_c,\ h,\ --}\right]\left[\frac{\partial (G,\ T_c,\ T_p,\cdot\ \cdot)}{\partial (G,\ T_c,\ h,\cdot\ \cdot)}\right]'$$

As mentioned in Section 4, the mass of the plate is divided equally between the three nodes. The variance of the mass included in the covariance matrix above is for one node only and hence in preparing the data use one-third the standard deviation of the mass of the total plate.

6.0 SYNTHESIS

6.1 Construction of the Mass and Stiffness Matrices

In Section 2.2.1 a truss example was used to describe the development of the stiffness matrix. It was noted that the first step in the stiffness matrix development was the construction of the undeleted stiffness matrix shown in Equation (2-8) and repeated in Equation (6-1) below,

$$[K]_{\text{undeleted}} = \left[\begin{array}{c|c|c|c} [1-1] & [1-2] & [1-3] & \cdot \\ \hline [2-1] & [2-2] & [2-3] & \cdot \\ \hline [3-1] & [3-2] & [3-3] & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right] \qquad (6\text{-}1)$$

where the submatrices $[i-j]$ are the nodal stiffness matrices.*

The undeleted mass matrix is a diagonal matrix as shown

$$[M]_{\text{undeleted}} = \left[\begin{array}{c|c|c|c} \lceil 1-1 \rfloor & [0] & [0] & \cdot \\ \hline [0] & \lceil 2-2 \rfloor & [0] & \cdot \\ \hline [0] & [0] & \lceil 3-3 \rfloor & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array}\right] \qquad (6\text{-}2)$$

The dimension of the undeleted mass or stiffness matrix is 6 x no. of nodes. The dimension of the dynamic mass or stiffness matrix is the dimension of the undeleted matrix less the number of restraints upon the system. To reduce the undeleted matrix, the rows and columns corresponding to each constrained degree of freedom are removed and the remaining columns and rows are shifted to fill in the voids to form the smaller matrix. This method only permits constraints that are aligned with the coordinates. Further sophistication of the constraints is possible but was not considered necessary in this analysis.

The mass and stiffness matrices of large dynamic models require a large amount of computer storage and as a result methods have been devised to manipulate the form and reduce the storage requirement. VIDAP has been designed to handle 300 degree-of freedom systems. The diagonal mass matrix is stored as a column

*Node numbers must be consecutive and start with 1.

and the symmetric and banded stiffness matrix is stored vertically as a semi-band in a rectangular format. These formats are shown pictorially in Figure 6-1.

System Mass Matrix VIDAP Form

$$\begin{bmatrix} m_{11} & & & & & \\ & m_{22} & & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & & m_{nn} \end{bmatrix} \Rightarrow \begin{bmatrix} m_{11} \\ m_{22} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ m_{nn} \end{bmatrix}$$

System Stiffness Matrix VIDAP Form

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & & 0 & \\ k_{21} & k_{22} & k_{23} & k_{24} & & \\ & \cdot & \cdot & \cdot & & \\ & & & & & \\ & 0 & & & & k_{nn} \end{bmatrix} \Rightarrow \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{22} & k_{23} & k_{24} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 \\ k_{nn} & 0 & 0 \end{bmatrix}$$

Figure 6-1 Storage of the System Mass and Stiffness Matrices in VIDAP

6.2 Compatibility

The purpose of VIDAP, as discussed earlier, is to produce statistical characteristics of eigenvalues and eigenvectors of large systems based upon the statistical characteristics of properties of individual structural components. This is accomplished by the development of partial derivatives used in a matrix chain as shown below

$$\left\{ {}^{(i)}\frac{d\lambda}{dx} \right\} = \left[\frac{\partial(\lambda,\ x)}{\partial(k)} \right] \Bigg| \left[\frac{\partial(\lambda,\ x)}{\partial(m)} \right] \left[{}^{(i)} \begin{matrix} \left[\dfrac{\partial(k)_r}{\partial(p)} \right] \\ \hline \left[\dfrac{\partial(m)}{\partial(p)} \right] \end{matrix} \right] \left\{ {}^{(i)}dp \right\}$$

$$\left[{}^{(i)}\frac{\partial(\lambda,\ x)}{\partial(k,\ m)_{syst}} \right] \left[{}^{(i)}\frac{\partial(k_r, m)}{\partial(p)} \right] \left\{ {}^{(i)}dp \right\} \qquad (6\text{-}3)$$

where the presuperscript $\left[{}^{(i)}\ \right]$ represents perturbations in the system due to perturbations in the properties of structural member, i.

The two large matrices in Equation (6-3) are developed from submatrices as shown. The two matrices $\left[\frac{\partial(k)_r}{\partial(p)}\right]$ and $\left[\frac{\partial(m)}{\partial p}\right]$ were developed in Section 4 and synthesized into $\left[\frac{\partial(k_r,m)}{\partial(p)}\right]$ in Subsections 4.1.5 and 4.2.6. The mass and stiffness elements are all oriented into system coordinates although the matrix element numbering is oriented to the local coordinate system.

The components of the matrices $\left[\frac{\partial(\lambda,x)}{\partial(k)_{syst}}\right]$, and $\left[\frac{\partial(\lambda,x)}{\partial(m)_{syst}}\right]$, are derived according to the methods developed in Section 3. These matrix components are developed, however, according to addresses in the system mass and stiffness matrices and it is here where a procedure must be developed to make compatible the partial derivatives in the two successive matrices of Equation (6-3).

Before continuing with the compatibility development, consider first the input and dimensional requirements. The vector $\left\{\frac{d\lambda}{dx}\right\}$ represents derivatives of all the eigenvalues and eigenvector components of interest to the program user. The user of VIDAP can select up to 100 eigenvalues and/or eigenvector components in several combinations to be evaluated statistically. The only restriction is that the selected components of any eigenvector must be in either one or two groups. For example if statistical characteristics of components of the jth eigenvector are desired, they must be obtained using one of the following options.

Eigenvector	Option 1	Option 2	Option 3
$\begin{Bmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \\ \vdots \\ x_{nj} \end{Bmatrix}$	statistical characteristics of all $\{x_j\}$	statistical characteristics of one section of $\{x_j\}$, i.e. consecutive elements x_{ij}, $x_{(i+1)j}$, $x_{(i+2)j}$, . . .	statistical characteristics of two sections of $\{x_j\}$ i.e. consecutive elements x_{ij}, $x_{(i+1)j}$, $x_{(i+2)j}$, . . . and x_{kj}, $x_{(k+1)j}$, $x_{(k+2)j}$, . . .

The number of rows and the notation of the rows in the matrix

$$\left[{}^{(i)}\frac{\partial(\lambda,x)}{\partial(k,m)}_{syst}\right]$$

correspond to the eigenvalue and eigenvector statistical data requested by the user. Hence the matrix will not exceed 100 rows (although the number of degrees of freedom of the system can far exceed this). The output of VIDAP will group together the data from the same eigenvalue and eigenvector. That is the successive rows of $\left[{}^{(i)}\frac{\partial(\lambda,x)}{\partial(k,m)}\right]$ will have λ_i, x_{1i}, x_{2i}, . . λ_j, x_{1j}, x_{2j}, etc.

Each column of $\left[{}^{(i)}\frac{\partial(\lambda,x)}{\partial(k,m)}_{syst}\right]$ has a single independent variable in all the partial derivatives whereas each row has a single dependent variable. This is shown below.

$$\left[{}^{(i)}\frac{\partial(\lambda,x)}{\partial(k,m)}_{syst}\right] = \begin{bmatrix} \frac{\partial\lambda_i}{\partial k_{rs}} & \frac{\partial\lambda_i}{\partial k_{r(s+1)}} & \frac{\partial\lambda_i}{\partial k_{r(s+2)}} & \cdot & \cdot & \cdot \\ \frac{\partial x_{1i}}{\partial k_{rs}} & \frac{\partial x_{1i}}{\partial k_{r(s+1)}} & \frac{\partial x_{1i}}{\partial k_{r(s+2)}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial\lambda_j}{\partial k_{rs}} & \frac{\partial\lambda_j}{\partial k_{r(s+1)}} & \frac{\partial\lambda_j}{\partial k_{r(s+2)}} & \cdot & \cdot & \cdot \\ \frac{\partial x_{1j}}{\partial k_{rs}} & \frac{\partial x_{1j}}{\partial k_{r(s+1)}} & \frac{\partial x_{1j}}{\partial k_{r(s+2)}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{6-4}$$

where the subscripts $(\ )_{rs}$ are in system coordinates.

The dependent variables in $\left[{}^{(i)}\frac{\partial(k_r,m)}{\partial(p)}\right]$ must correspond to the independent variables in $\left[{}^{(i)}\frac{\partial(\lambda,x)}{\partial(k,m)}_{syst}\right]$ the problem is that the coordinates of the structural element may be found in a number of different locations in the structural coordinates. For example consider a beam element with end points at node 3 and node 5 of a six-node system. Let each node have three restraints. The symmetrical element and system matrices will be as shown in Figure 6-2.

The stiffnesses of the particular beam are not the only stiffnesses located in the addresses shown in Figure 6-2. However, if this beam joining nodes 3 and 5 is allowed to vary while all of the rest of the system remains constant, then the derivatives associated with these addresses will be exclusively those of this particular beam element.

If the address of a stiffness element is known in the system stiffness matrix, the appropriate partial derivatives can be developed according to the methods of Section 3. To do this an accounting procedure must be used to find the locations of the beam stiffness elements in the system stiffness matrix.

The procedure is as follows:

(1) The two nodes of the beam are treated in ascending order, i.e. 3 comes before 5.

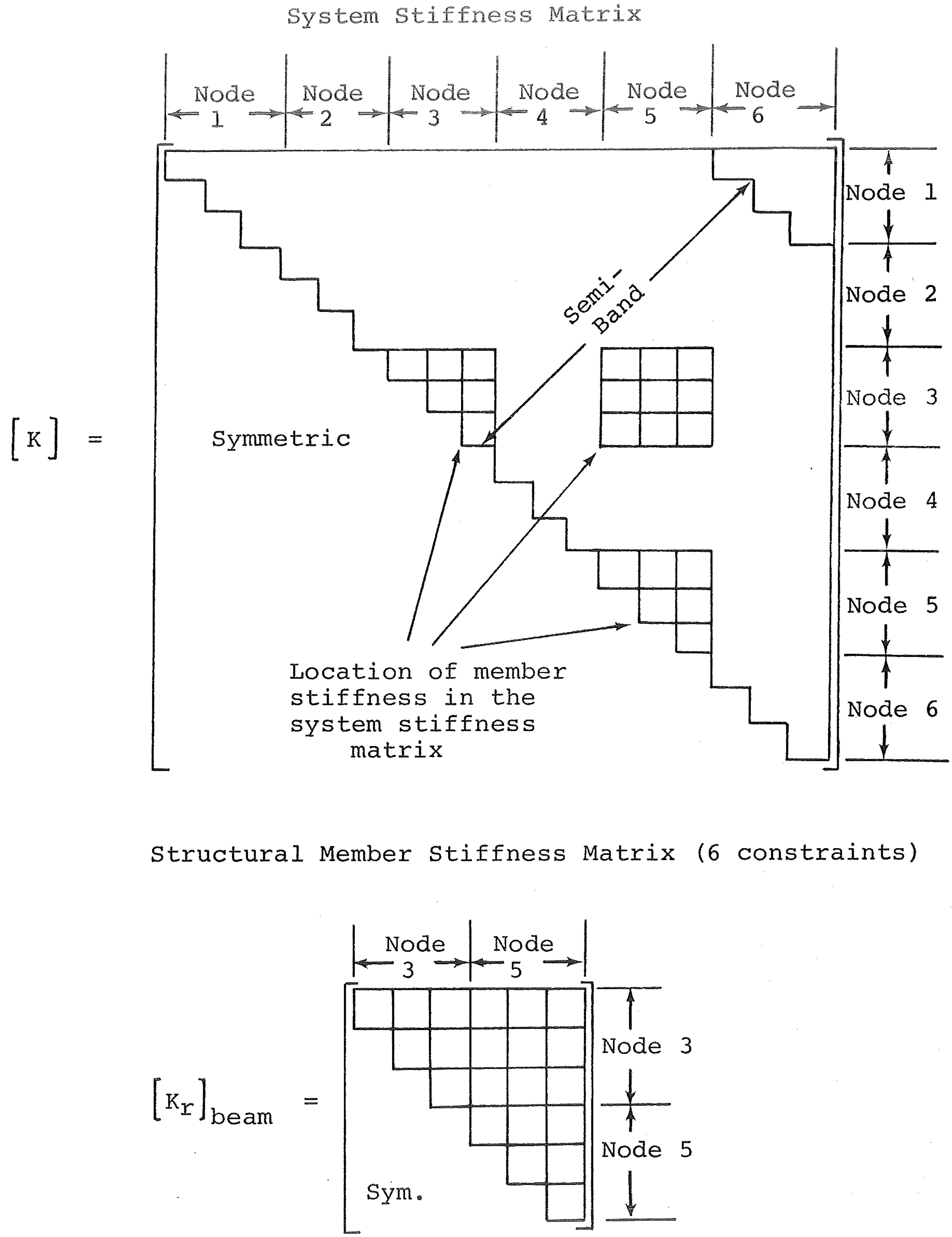


Figure 6-2 Beam Element Stiffness Locations

(2) The number of unconstrained coordinates are counted in all the nodes prior to the lowest node of the beam. In this case there are 6 unconstrained coordinates before node 3. The sum of these coordinates gives the number used to determine the first row in the system stiffness matrix.

(3) The number of unconstrained coordinates in the first beam node are counted and this number is used to establish the number of consecutive rows in the stiffness matrix which are occupied by the stiffness of this first node.

(4) The number of unconstrained coordinates between the two beam nodes are counted to establish the row number in the system matrix corresponding to the stiffnesses of the second node of the beam.

(5) The number of unconstrained coordinates in the second beam node are counted and this number establishes the consecutive rows in the stiffness matrix occupied by the stiffnesses of the second beam node.

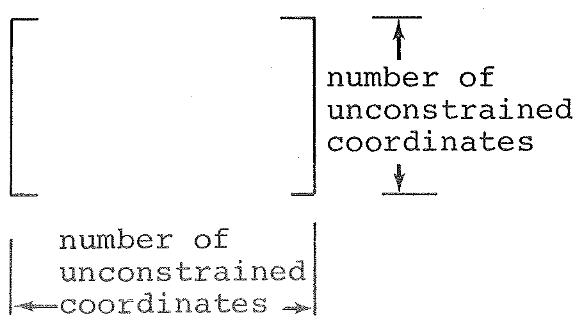
To implement this procedure two new matrices are introduced to the analysis. These are called [KR] and [KS] and are strictly internal to the program.

[KR] and [KS] are square matrices having the same number of rows and columns as there are unconstrained coordinates in the member (in this case, the beam).

Each row of [KR] has only a single number corresponding to a row in the system stiffness matrix. Each column of [KS] has only a single number and these correspond to single columns in the system stiffness matrix. The development and implementation of these two matrices is as follows:

(a) From steps (3) and (5) above, count the number of unconstrained coordinates in the beam and size [KR] accordingly

[KR] =



(b) From step (2) find the number of the row containing the first member stiffness in the system matrix and enter this number in the first row of [KR].

$$[KR] = \begin{bmatrix} r & r & r & r & r & r & \cdot & \cdot \\ & & & & & & & \\ & & & & & & & \end{bmatrix}$$

(c) From step (3) take the number of unconstrained coordinates in the node and number the same number of rows in [KR] consecutively.

$$[KR] = \begin{bmatrix} r & r & r & r & r & \cdot \\ r+1 & r+1 & r+1 & r+1 & r+1 & \cdot \\ r+2 & r+2 & r+2 & r+2 & r+2 & \cdot \\ & & & & & \end{bmatrix}$$

(d) From steps (2), (3), and (4) count the total number of rows in the system stiffness matrix preceeding the stiffness of the second node of the member. Let s be the row number corresponding to the first row in the stiffness matrix of this second node.

$$[KR] = \begin{bmatrix} r & r & r & \cdot & \cdot \\ r+1 & r+1 & r+1 & \cdot & \cdot \\ r+2 & r+2 & r+2 & \cdot & \cdot \\ s & s & s & \cdot & \cdot \\ & & & & \end{bmatrix}$$

(e) Complete [KR] by numbering the remaining rows consecutively, s+1, s+2, etc.

$$[KR] = \begin{bmatrix} r & r & r & \cdot & \cdot \\ r+1 & r+1 & r+1 & \cdot & \cdot \\ r+2 & r+2 & r+2 & \cdot & \cdot \\ s & s & s & \cdot & \cdot \\ s+1 & s+1 & s+1 & \cdot & \cdot \\ s+2 & s+2 & s+2 & \cdot & \cdot \end{bmatrix}$$

Look now at $[K_r]_{beam}$ in Figure 6-3. If we move across the first row, then start again at the diagonal and complete the second row and so on we can form a column of stiffness elements without duplication because of symmetry.

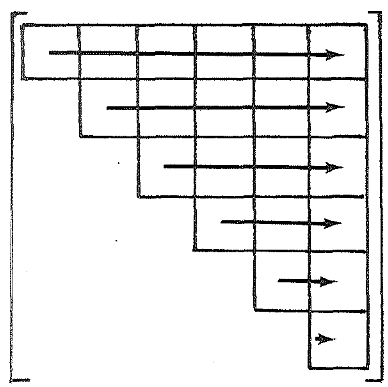
The partial derivatives in the matrix $\left[\frac{\partial (k)_r}{\partial (p)}\right]$, Equation (4-3), are ordered vertically in this way. Thus if we form a column of indices from [KR] and [KS] by going across the rows of [KR] and [KS] in the same way, we will have a set of indices for the system coordinates which are in a column compatible with the partial derivatives of the member stiffnesses. The column of indices is shown below.

$$\{\text{ind.}\} = \begin{Bmatrix} KR_{11},\ KS_{11} \\ KR_{12},\ KS_{12} \\ \cdot \\ \cdot \\ KR_{22},\ KS_{22} \\ KR_{23},\ KS_{23} \\ \cdot \\ \cdot \\ KR_{nn},\ KS_{nn} \end{Bmatrix} \equiv \underbrace{\begin{Bmatrix} r,\ r \\ r,\ r+1 \\ r,\ r+2 \\ \cdot \\ r+1,\ r+1 \\ r+1,\ r+2 \\ r+1,\ r+3 \\ \cdot \\ s+2,\ s+2 \end{Bmatrix} = \begin{Bmatrix} 7,\ 7 \\ 7,\ 8 \\ 7,\ 9 \\ \cdot \\ 8,\ 8 \\ 8,\ 9 \\ 8,\ 10 \\ \cdot \\ 15,\ 15 \end{Bmatrix}}_{\text{Samples discussed on the previous pages}}$$

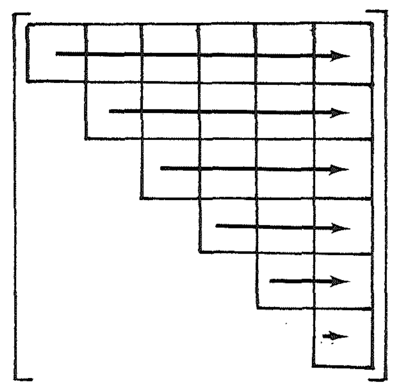
The vector, {ind.}, provides the indices associated with k_{rs} and m_{rs} in computing the partial derivatives for eigenvalues and eigenvectors in Equations (3-8), (3-9), (3-29), and (3-30). This entire procedure is pictured in the flow diagram in Figure 6-4.

Because only the diagonal and one triangular half of the matrix $\frac{\partial}{\partial p_j}[K]$ (see Equations (4-2,3)) are used in developing $\left[\frac{\partial (k)_r}{\partial (p)}\right]$, some compensation must be made for the omitted derivatives before the final matrix multiplication takes place. This is accomplished by putting into the spaces occupied by the partial derivatives having r = s, the sum $\frac{\partial (\ )}{\partial (\ )_{rs}} + \frac{\partial (\ )}{\partial (\ )_{sr}}$, i.e. $\frac{\partial \lambda_i}{\partial k_{rs}} + \frac{\partial \lambda_i}{\partial k_{sr}}$ replaces $\frac{\partial \lambda_i}{\partial k_{rs}}$.

$[K_r]_{beam} =$



$[KR] =$



$[KS] =$

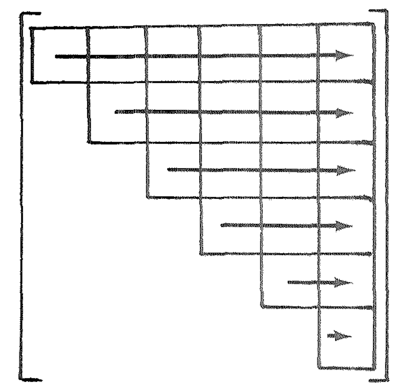


Figure 6-3 Order of Development of Partial Derivatives

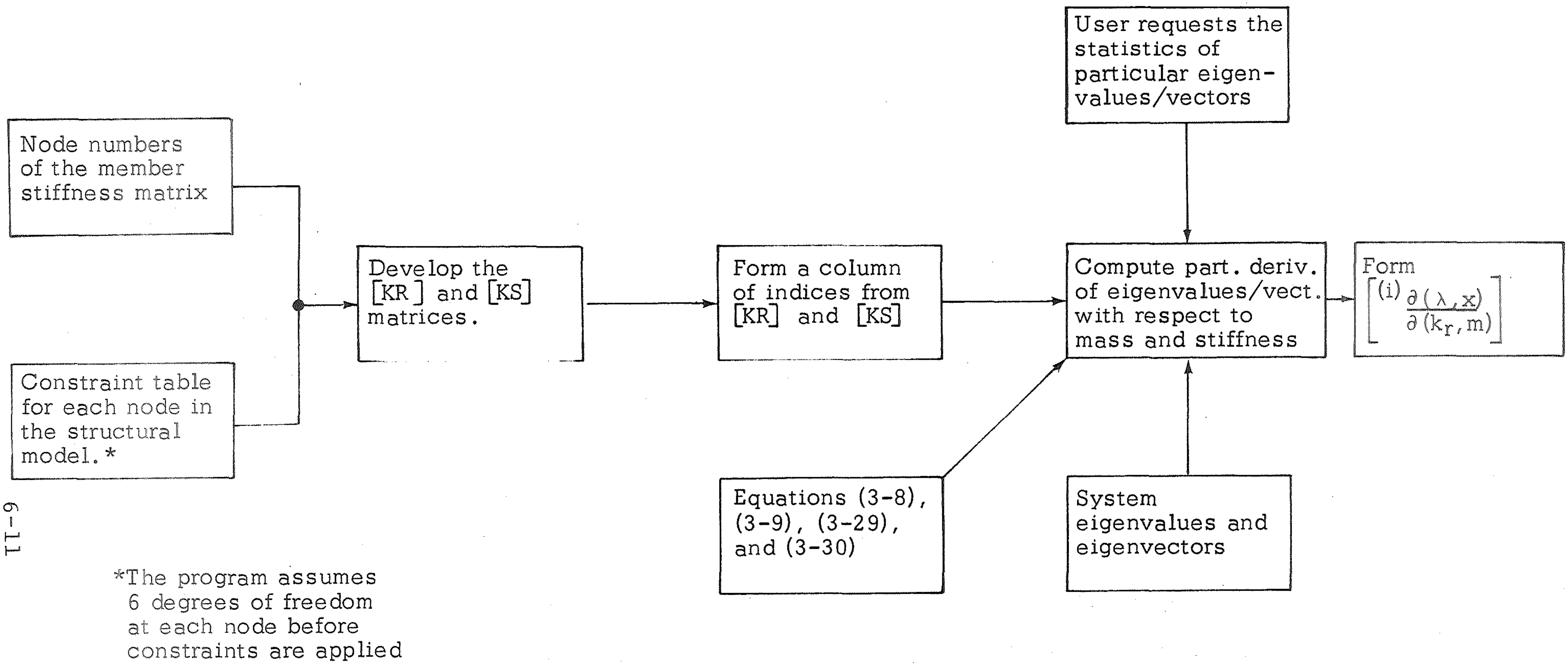


Figure 6-4 Compatibility Procedure for the Development of Eigenvalue/Vector Partial Derivatives

In the case of the beam element in Figure 6-2, [KR] is

$$[KR] = \begin{bmatrix} 7 & 7 & 7 & 7 & 7 & 7 \\ 8 & 8 & 8 & 8 & 8 & 8 \\ 9 & 9 & 9 & 9 & 9 & 9 \\ 13 & 13 & 13 & 13 & 13 & 13 \\ 14 & 14 & 14 & 14 & 14 & 14 \\ 15 & 15 & 15 & 15 & 15 & 15 \end{bmatrix}$$

Since the stiffness matrix is symmetrical, [KS] is the transpose of [KR].

$$[KS] = \begin{bmatrix} r & r+1 & r+2 & s & s+1 & s+2 \\ r & r+1 & r+2 & s & s+1 & s+2 \\ r & r+1 & r+2 & s & s+1 & s+2 \\ r & r+1 & r+2 & s & s+1 & s+2 \\ r & r+1 & r+2 & s & s+1 & s+2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

or in the example

$$[KS] = \begin{bmatrix} 7 & 8 & 9 & 13 & 14 & 15 \\ 7 & 8 & 9 & 13 & 14 & 15 \\ 7 & 8 & 9 & 13 & 14 & 15 \\ 7 & 8 & 9 & 13 & 14 & 15 \\ 7 & 8 & 9 & 13 & 14 & 15 \\ 7 & 8 & 9 & 13 & 14 & 15 \end{bmatrix}$$

Note that [KR] and [KS] have the same dimensions as the member stiffness matrix and contain all the address information about the member stiffnesses in the system stiffness matrix.

With [KR] and [KS] it is now a relatively simple matter to develop the proper partial derivatives for $\left[{}^{(i)}\frac{\partial(\lambda,x)}{\partial(k,m)}_{syst} \right]$.

Thus the subscripts in the vector {ind.} are used in both orders. A typical row in $\left[{}^{(i)}\dfrac{\partial(\lambda,x)}{\partial(k_r,m)} \right]$ would then appear as shown below.

$$\begin{bmatrix} \dfrac{\partial\lambda_i}{\partial k_{11}} & \dfrac{\partial\lambda_i}{\partial k_{12}} + \dfrac{\partial\lambda_i}{\partial k_{21}} & \dfrac{\partial\lambda_i}{\partial k_{13}} + \dfrac{\partial\lambda_i}{\partial k_{31}} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The expression $\dfrac{\partial\lambda_i}{\partial k_{rs}}$ is symmetrical and thus

$$\frac{\partial\lambda_i}{\partial k_{rs}} + \frac{\partial\lambda_i}{\partial k_{sr}} = 2\frac{\partial\lambda_i}{\partial k_{rs}} . \qquad \frac{\partial x_{ji}}{\partial k_{rs}} \text{ is not symmetrical}$$

however, but the sum has been derived as shown in Equation (3-30).

6.3 The Solution of Eigenvector Partial Derivatives

The previous section described how compatibility relations were used for selecting the proper indices for eigenvalue/vector partial derivatives. It was pointed out how these indices were used with certain equations in Section 3 to compute and order the partial derivatives. The purpose of this section is to describe breifly the operations involved in the development of partial derivatives for the eigenvectors. Whereas the expressions for the eigenvalue partial derivatives, Equations (3-8) and (3-9), are quite straightforward, those for the eigenvectors are more complex and involve considerable computation. In fact, the eigenvector partial derivative computation involves the largest single operation in the VIDAP program.

The eigenvector partial derivatives are developed in two steps:

(1) One element of the eigenvector is held fixed while partial derivatives are developed for the other elements relative to the fixed element. The equations for partials with respect to diagonal and off-diagonal elements in the stiffness matrix are (from Section 3):

$$\left\{\frac{\partial \overline{x_i}^u}{\partial k_{rr}}\right\} = \left[\overline{F_i}^u\right]^{-1} \left(\frac{x_{ri}^{\;2}}{x_i^T M x_i} \left\{\overline{Mx_i}^u\right\} - x_{ri}\left\{\overline{\delta_{jr}}^u\right\}\right)^* \tag{6-5}$$

$$\left\{\frac{\partial \overline{x_i}^u}{\partial k_{rs}}\right\} + \left\{\frac{\partial \overline{x_i}^u}{\partial k_{sr}}\right\} = \left[\overline{F_i}^u\right]^{-1} \left(\frac{2x_{ri}x_{si}}{x_i^T M x_i} \left\{\overline{Mx_i}^u\right\} - \left\{\overline{x_{si}\delta_{jr} + x_{ri}\delta_{js}}^u\right\}\right)^* \tag{6-6}$$

(2) The partial derivatives developed in Step (1) are modified to maintain constant generalized mass and in so doing the fixed element is permitted to take on a non-zero value. The expression is:

$$\frac{\partial x_{pi}}{\partial k_{rs}} = \frac{\partial x_{pi}}{\partial k_{rs}}^{u} - \frac{x_{pi}}{M_i} \left\{\frac{\partial x_i}{\partial k_{rs}}^{u}\right\} [M]\{x_i\} \quad {**}$$

---

*$\{^{-u}\}$ represents omission of the uth element, $[^{-u}]$ represents omission of the uth row and column. In VIDAP u = 0, but can be extended to operate for any value.

**$\{\,^{u}\}$ represents represents replacement of the uth element by a zero. Hence a vector $\{\,^{u}\}$ has n elements whereas $\{^{-u}\}$ has n-1 elements.

In the interest of minimizing the computer storage, the matrix $\left[\bar{F}_i^{u}\right]$, which is symmetrical, is formed from $[K]$, $[M]$, and λ_i and stored in a semi-band. The operation is shown in Figure 6-5. $\left[\bar{F}_i^{u}\right]$, which in its symmetrical form is (n-1)x(n-1), is also of order n-1 and thus capable of inversion.

The number of operations in the solution of Equation (6-5) or (6-6) can be minimized by returning $\left[\bar{F}_i^{u}\right]$ to the left-hand side of the equation and by solving the column of equations simultaneously rather than by inverting $\left[\bar{F}_i^{u}\right]$. A decomposition procedure involving banded matrices was selected for the solution of (6-5) and (6-6). The decomposition method (Ref. 7) breaks $\left[\bar{F}_i^{u}\right]$ into the product of three matrices.

$$\left[\bar{F}_i^{u}\right] = [S]'[D][S] \qquad (6\text{-}7)$$

where $[S]$ is an upper triangular matrix with unit diagonal and $[D]$ is a diagonal but not unitary matrix.

The decomposition procedure as outlined in Reference 7 permits the matrix $\left[\bar{F}_i^{u}\right]$ to have negative numbers on the diagonal. Other decomposition methods (Ref. 4) which use only the produce $[S]'[S]$ (with the diagonal non-unitary), incorporate a square-root operation in the solution which will not work with the negative diagonal elements.

The second step of the partial derivative development involves no special operations and therefore does not require further elaboration.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdot \\ k_{22} & k_{23} & k_{24} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot \\ k_{nn} & 0 & 0 & \cdot \end{bmatrix} - \lambda_i \begin{bmatrix} m_{11} & 0 & \cdot & \cdot \\ m_{22} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_{nn} & 0 & \cdot & \cdot \end{bmatrix}$$

Input [M], [K], and λ_i

$$\Downarrow$$

$$[F_i] = \begin{bmatrix} k_{11}-\lambda_i m_{11} & k_{12} & k_{13} & \cdot \\ k_{22}-\lambda_i m_{22} & k_{23} & k_{24} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 \\ k_{nn}-\lambda_i m_{nn} & 0 & 0 & 0 \end{bmatrix}$$

Form $[F_i]$

$$\Downarrow$$

$$[\bar{F}_i^u] = \begin{bmatrix} k_{11}-\lambda_i m_{11} & k_{12} & \cdot & \cdot & k_{1(u-1)} & k_{1(u+1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{(u-1)(u-1)}-\lambda_i m_{(u-1)(u-1)} & k_{(u-1)(u+1)} & \cdot & \cdot & \cdot & \cdot \\ k_{(u+1)(u+1)}-\lambda_i m_{(u+1)(u+1)} & k_{(u+1)(u+2)} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{nn}-\lambda_i m_{nn} & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

Form $[\bar{F}_i^u]$

Figure 6-5 Formation of Banded $[\bar{F}_i^u]$

6.4 Program Operation

The operation of VIDAP is summarized in the flowchart in Fig. 6-6. The following explanation summarizes the operations in each step and refers the reader to other sections of the report for elaboration where necessary.

Step 1

As input to the program, the user provides the structural stiffness matrix, the mass matrix, eigenvalues and eigenvectors of the modes of interest, the number of nodes in the structure, the physical locations of the nodes, and a list of coordinate constraints for each node.

Step 2

The user requests statistics of the frequencies and modes of interest. The total number of eigenvalue and eigenvector components can range upwards to 100.

Step 3

Beam and plate property data are entered for the development of the partial derivatives. Property covariance matrices are input as 9 x 9 matrices. The node numbers must be provided for each random member.

Step 4

If an arbitrary stiffness matrix is used, a covariance matrix for stiffness and mass properties must be developed outside the program. The external development of this $\left[{}^{(i)}\sum_{k_r,m}\right]$ removes the need for the property covariance matrix and the development of $\left[{}^{(i)}\frac{\partial(k_r,m)}{\partial(p)}\right]$. $[KR]$ and $[KS]$ must be computed externally in this case and provided as input data.

Step 5

Program operation begins when the question is asked about the type of element. If it is a beam or plate, the program digests the data in Step 3 and proceeds to compute $\left[{}^{(i)}\frac{\partial(k_r,m)}{\partial(p)}\right]$ in Step 6. If it is an arbitrary stiffness matrix, the program bypasses Steps 6 and 7 and uses the data provided in Step 4.

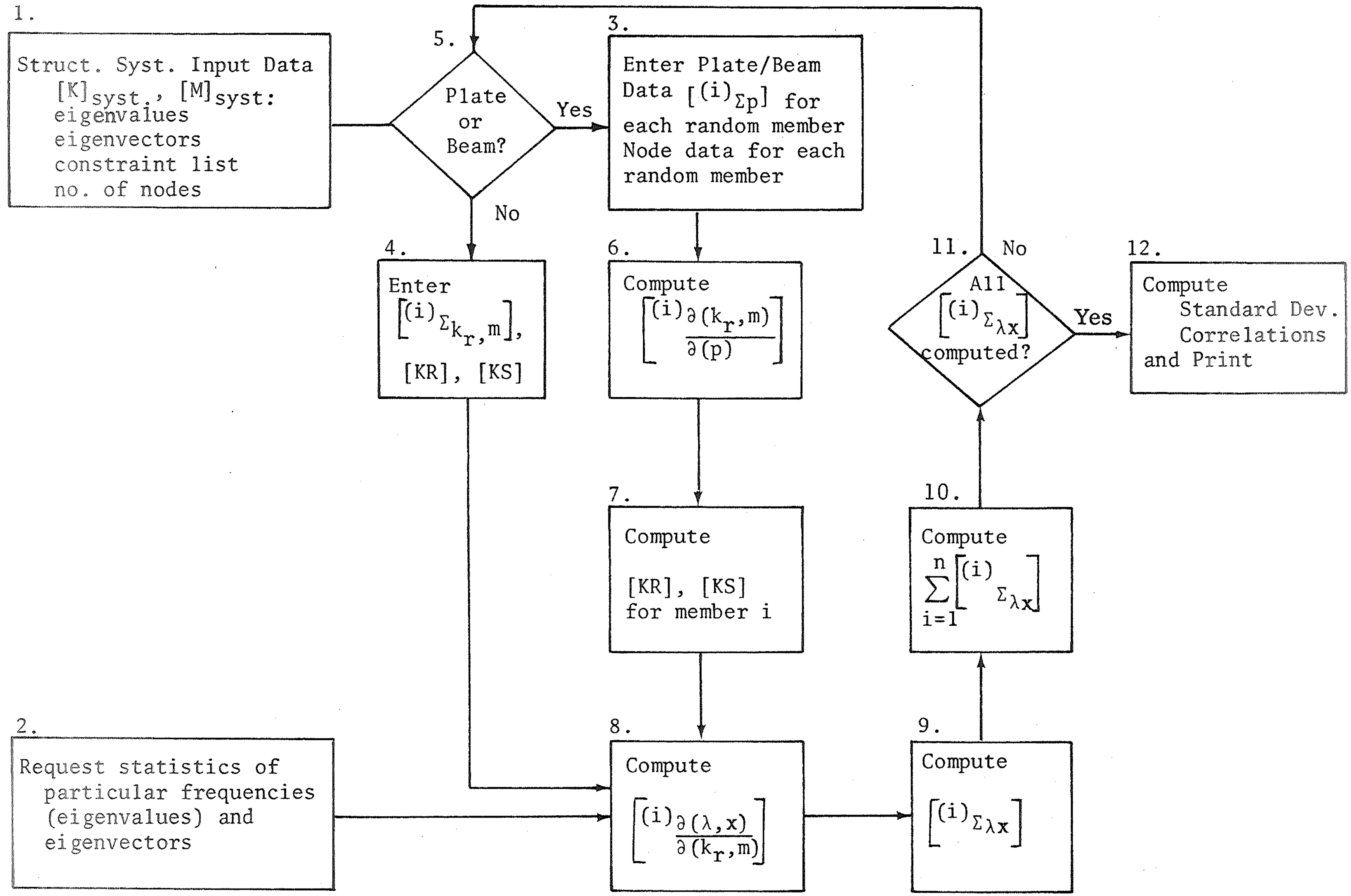


Figure 6-6 Basic VIDAP Operations

Step 6

With the data from Steps 1 and 3, the program develops $\left[{}^{(i)} \dfrac{\partial (k_r, m)}{\partial (p)} \right]$ for the first random structural member.

Step 7

Using data from Steps 1 and 3, the program develops $[KR]$ and $[KS]$ for the first random structural member.

Step 8

The user output request in Step 2, the specified eigenvalues and eigenvectors, the $[K]_{syst}$ and $[M]_{syst}$, and $[KR]$ and $[KS]$ are employed as described in Sections 6.2 and 6.3 to form $\left[{}^{(i)} \dfrac{\partial (\lambda, x)}{\partial (k_r, m)} \right]$ for the first random structural member.

Step 9

The matrices developed in Steps 6 and 8 and entered in Steps 3 or 4 are multiplied together in the products shown below to obtain the covariance matrix for the eigenvalues and eigenvectors.

For a beam or plate

$$\left[{}^{(i)} \sum_{\lambda x} \right] = \left[{}^{(i)} \frac{\partial (\lambda, x)}{\partial (k_r, m)} \right] \left[{}^{(i)} \frac{\partial (k_r, m)}{\partial (p)} \right] \left[{}^{(i)} \sum_{p} \right] \times \left[{}^{(i)} \frac{\partial (k_r, m)}{\partial (p)} \right]' \left[{}^{(i)} \frac{\partial (\lambda, x)}{\partial (k_r, m)} \right]'$$

For an arbitrary stiffness matrix

$$\left[{}^{(i)} \sum_{\lambda x} \right] = \left[{}^{(i)} \frac{\partial (\lambda, x)}{\partial (k_r, m)} \right] \left[{}^{(i)} \sum_{k_r, m} \right] \left[{}^{(i)} \frac{\partial (\lambda, x)}{\partial (k_r, m)} \right]'$$

Step 10

The eigenvalue/vector covariance matrices resulting from each random member are summed into a single covariance matrix.

Step 11

The covariance matrix computed in Step 9 is stored, and the procedure starting in Step 5 is repeated for each random structural member.

Step 12

The covariance matrix resulting from Step 11 is used to compute the standard deviations and the correlation matrix of all the requested output data. Equation (2-10) is used to transform standard deviations of eigenvalues into standard deviations of frequencies. The final output contains mean values and standard deviations of eigenvalues, frequencies, and eigenvector components.

7.0 EXAMPLES AND VERIFICATION

7.1 Methodology

Verification of the linear statistical model can be made in two ways which are quite independent of each other. The first method involves developing partial derivatives numerically by perturbing a sample problem and comparing these partial derivatives with those computed within the program. The second method is to use the Monte Carlo method to generate the statistics of a sample system and to compare the results with the statistical characteristics generated by VIDAP.

The above methods were used in evaluating two sample problems: a simple longitudinal rod and an SII longitudinal vibration analysis provided by the George C. Marshall Space Flight Center.*

The selection of the four degree-of-freedom longitudinal rod was convenient because this problem had been thoroughly checked earlier in the development of the J. H. Wiggins Company PASS I (Probabilistic Analysis of Structural Systems) Program. Thus, the partial derivatives developed by a proven program could be compared with those developed by VIDAP. The SII longitudinal vibration problem gave the opportunity to demonstrate VIDAP on a larger system and in an area of interest to NASA.

7.2 Four Degree-of-Freedom Longitudinal Vibrations

The longitudinal rod studied in this example is shown below in Fig. 7-1.

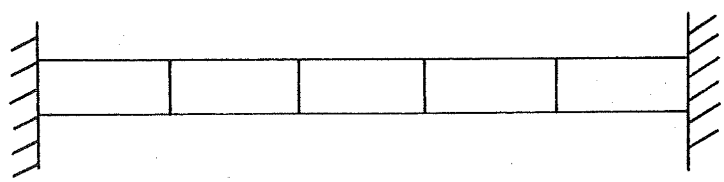


Figure 7-1 Four Degree-of-Freedom Longitudinal Fixed-Fixed Rod

The mass and stiffness matrices are as follows. (Note that the mass matrix is non-diagonal and is based upon the consistent mass matrix approach described in Reference 8.)

*Neither of these problems contain the plate or beam elements discussed in Section 4 since the checkout of this section of VIDAP was not successfully completed at the writing of this report.

$$[M] = \frac{1}{6}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The stiffness matrix is composed of five stiffnesses, each relating to one of the five finite elements shown in Fig. 7-1.

$$[K] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix}$$

k_1, k_2, k_3, k_4, and k_5 are assumed to be independent random variables, each having a standard deviation, $\sigma = .05$. Masses are assumed to be fixed.

The stiffnesses k_1, k_2, ... can be treated as the properties of the system. A matrix equivalent to $\left[\frac{\partial (k,m)}{\partial (p)}\right]$ can be formed as follows:

$$\begin{Bmatrix} dk_{11} \\ dk_{12} \\ dk_{13} \\ dk_{14} \\ dk_{22} \\ dk_{23} \\ dk_{24} \\ dk_{33} \\ dk_{34} \\ dk_{44} \\ dm_{11} \\ dm_{22} \\ dm_{33} \\ dm_{44} \end{Bmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{Bmatrix} dk_1 \\ dk_2 \\ dk_3 \\ dk_4 \\ dk_5 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

or

$$\left\{dk_{ij}, dm_{ii}\right\} = \left[\frac{\partial(k_{ij}, m_{ii})}{\partial(k_\ell)}\right]\left\{dk_\ell\right\}$$

$\left[\frac{\partial(k_{ij}, m_{ii})}{\partial(k_\ell)}\right]$ has nine columns which make it compatible with VIDAP.* Note that all the stiffness elements on and above the diagonal are considered. However, only the diagonal mass elements are considered because in the design of the program it was anticipated that most mass matrices would be diagonal, and the partial derivative matrices could have small dimensions if mass partial derivatives were restricted to the diagonal.

---

*VIDAP permits the input of nine random properties. The A(I,J) matrix (equivalent to $\left[\frac{\partial(k_{ij}, m_{ii})}{\partial(k_\ell)}\right]$) in step 18 of Appendix B must always have nine columns even though, as in this case, all nine are not necessary.

The covariance matrix for this system of five independent random variables has the variances of the five successive k_i's down the diagonal. The remaining four zeros on the diagonal account for the remaining four random variables which are available in VIDAP but not used in this problem.

$$\left[\Sigma_k\right] = \underbrace{\begin{bmatrix} (.05)^2 & 0 & 0 & 0 & 0 & . & 0 \\ 0 & (.05)^2 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & (.05)^2 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & (.05)^2 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & (.05)^2 & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{9 columns}}$$

The $[KR]$ matrix is (see Section 6.2)

$$[KR] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$[KS]$ is the transpose of $[KR]$.

The results of the VIDAP run are shown in Tables 7-1 and 7-2. The statistics of the eigenvalues compare exactly with those of PASS I, and the ratio of the standard deviation of the frequency to the frequency is a constant as would be expected by a chain of uncorrelated stiffnesses (Reference 9). The statistics of the eigenvectors do not agree with the PASS I output because PASS I fixes the first element in each eigenvector, thus making the standard deviation of this element zero and changing the other elements in a manner discussed in Section 3.

The eigenvector partial derivative development in VIDAP is in two steps, the first of which holds the first element of the eigenvector constant in a manner analogous to PASS I.

The equations leading to the partial derivatives in PASS I are different (see Reference 1), but the results should be the same. The first-stage partial derivatives developed by VIDAP were compared with those developed by PASS I, and the two programs did indeed agree to four significant figures. The second stage of the partial derivative development was then hand computed, and the answers agreed with the VIDAP output.

A set of independent perturbations were made of elements in the stiffness matrix to determine the range of linearity of the partial derivatives. Figures 7-2,3,4 and 5 show per cent changes in specific eigenvector elements vs. per cent changes in the stiffness element k_{11}. It is evident that, for this system at least, the assumption of linearity is reasonable for changes in k_{11} of 10 to 20 per cent.

TABLE 7-1 STATISTICAL CHARACTERISTICS OF A

LONGITUDINAL FIXED-FIXED SYSTEM

MODE	EIGENVALUE	STANDARD DEVIATION	FREQUENCY	STANDARD DEVIATION
1	4.07935600E-01	1.11717775E-02	1.01652003E-01	1.39192755E-03
2	1.79525000E+00	4.91724713E-02	2.13246841E-01	2.92044966E-03
3	4.64469500E+00	1.27200236E-01	3.43003648E-01	4.69677195E-03
4	9.11356400E+00	2.49585291E-01	4.80467773E-01	6.57907777E-03

VECTOR	EIGENVECTOR	STANDARD DEVIATION
X(1, 1)	3.84177700E-01	1.24574697E-02
X(1, 2)	6.21612500E-01	5.53524921E-03
X(1, 3)	6.21612500E-01	5.53525059E-03
X(1, 4)	3.84177700E-01	1.24574666E-02
X(2, 1)	-6.85620000E-01	1.22409384E-02
X(2, 2)	-4.23736300E-01	1.92831770E-02
X(2, 3)	4.23736300E-01	1.92852831E-02
X(2, 4)	6.85620000E-01	1.22419785E-02
X(3, 1)	8.01174800E-01	1.41059591E-02
X(3, 2)	-4.95153200E-01	3.64803819E-02
X(3, 3)	-4.95153200E-01	3.64803843E-02
X(3, 4)	8.01174800E-01	1.41059565E-02
X(4, 1)	-5.90006000E-01	2.65379792E-02
X(4, 2)	9.54650000E-01	1.84832667E-02
X(4, 3)	-9.54650000E-01	1.84832720E-02
X(4, 4)	5.90006000E-01	2.65379696E-02

TABLE 7-2 CORRELATION MATRIX OF A RANDOM

LONGITUDINAL FIXED-FIXED SYSTEM

λ_1, ω_1	1.00000000E+00					
x(1,1)	-1.90742684E-01	1.00000000E+00				
x(1,2)	2.65309486E-01	1.97233408E-01	1.00000000E+00			
x(1,3)	2.65309496E-01	-8.04502635E-01	-1.55331851E-01	1.00000000E+00		
x(1,4)	-1.90742642E-01	-5.63407543E-01	-8.04502513E-01	1.97233569E-01	1.00000000E+00	
λ_2, ω_2	6.66667009E-01	-1.90742603E-01	2.65309374E-01	2.65309383E-01	-1.90742561E-01	1.00000000E+00
x(2,1)	2.43185613E-01	-1.83517815E-01	-5.10751711E-01	5.10751552E-01	1.83517883E-01	-2.43185273E-01
	1.00000000E+00					
x(2,2)	-2.49771627E-01	6.10505456E-01	5.86685700E-01	-5.86685630E-01	-6.10505551E-01	2.49771279E-01
	-8.07171196E-01	1.00000000E+00				
x(2,3)	2.49800391E-01	6.10449296E-01	5.86650643E-01	-5.86650734E-01	-6.10449275E-01	-2.49800042E-01
	-7.82275032E-02	2.51283671E-01	1.00000000E+00			
x(2,4)	-2.43198657E-01	-1.83512457E-01	-5.10736564E-01	5.10736562E-01	1.83512412E-01	2.43198318E-01
	2.90291288E-01	-7.82484597E-02	-8.07208056E-01	1.00000000E+00		
λ_3, ω_3	6.66666529E-01	1.90742455E-01	-2.65309168E-01	-2.65309177E-01	1.90742413E-01	3.33333599E-01
	2.43185720E-01	-2.49771738E-01	2.49800502E-01	-2.43198765E-01	1.00000000E+00	

TABLE 7-2 (Cont'd)

x(3,1)					
8.07754873E-02	-6.70218239E-01	1.28582573E-01	1.28583418E-01	4.85330439E-01	8.07754802E-02
-4.31274679E-01	-1.69324274E-01	-1.69281281E-01	-4.31261892E-01	-8.07755262E-02	1.00000000E+00
x(3,2)					
5.05371341E-02	-5.78377569E-02	-4.22350574E-01	5.83246100E-01	-5.78372041E-02	5.05371296E-02
7.69970525E-01	-3.82887456E-01	-3.82889543E-01	7.69947848E-01	-5.05371584E-02	-7.00507686E-01
1.00000000E+00					
x(3,3)					
5.05370962E-02	-5.78372344E-02	5.83246129E-01	-4.22350473E-01	-5.78376399E-02	5.05370917E-02
-7.69970560E-01	3.82887454E-01	3.82889458E-01	-7.69947803E-01	-5.05371205E-02	7.49493664E-01
-9.69351998E-01	1.00000000E+00				
x(3,4)					
8.07754676E-02	4.85330412E-01	1.28583233E-01	1.28582398E-01	-6.70218238E-01	8.07754605E-02
4.31274620E-01	1.69324341E-01	1.69281218E-01	4.31261959E-01	-8.07755065E-02	-9.21703862E-01
7.49493630E-01	-7.00507771E-01	1.00000000E+00			
λ_4, ω_4					
3.33333144E-01	1.90742483E-01	-2.65309208E-01	-2.65309217E-01	1.90742442E-01	6.66666620E-01
-2.43185474E-01	2.49771485E-01	-2.49800249E-01	2.43198519E-01	6.66666586E-01	-8.07755112E-02
-5.05371491E-02	-5.05371112E-02	-8.07754915E-02	1.00000000E+00		
x(4,1)					
-1.30367805E-01	-7.96450662E-02	4.26959969E-01	-4.26959782E-01	7.96450281E-02	1.30367604E-01
-9.03505743E-01	4.97167445E-01	1.06406645E-01	-5.23031768E-01	-1.30367874E-01	7.48098827E-01
-9.36094885E-01	9.36094927E-01	-7.48098795E-01	1.30367701E-01	1.00000000E+00	
x(4,2)					
-1.15683481E-01	1.85027027E-01	-3.55662243E-01	3.55662160E-01	-1.85027076E-01	1.15683303E-01
5.13967524E-01	-4.32502189E-02	-3.90018682E-01	8.51546984E-01	-1.15683542E-01	-8.20545173E-01
9.28701311E-01	-9.28701315E-01	8.20545216E-01	1.15683389E-01	-8.11245320E-01	1.00000000E+00
x(4,3)					
1.15683504E-01	1.85027034E-01	-3.55662312E-01	3.55662123E-01	-1.85027007E-01	-1.15683325E-01
8.51558106E-01	-3.89983536E-01	-4.32454774E-02	5.13938293E-01	1.15683565E-01	-8.20545178E-01
9.28701281E-01	-9.28701328E-01	8.20545151E-01	-1.15683411E-01	-9.92222052E-01	8.39408038E-01
1.00000000E+00					
x(4,4)					
1.30367750E-01	-7.96450270E-02	4.26959929E-01	-4.26959862E-01	7.96450749E-02	-1.30367549E-01
-5.23063130E-01	1.06421568E-01	4.97197549E-01	-9.03494856E-01	1.30367819E-01	7.48098823E-01
-9.36094954E-01	9.36094947E-01	-7.48098870E-01	-1.30367646E-01	7.96051015E-01	-9.92222050E-01
-8.11245373E-01	1.00000000E+00				

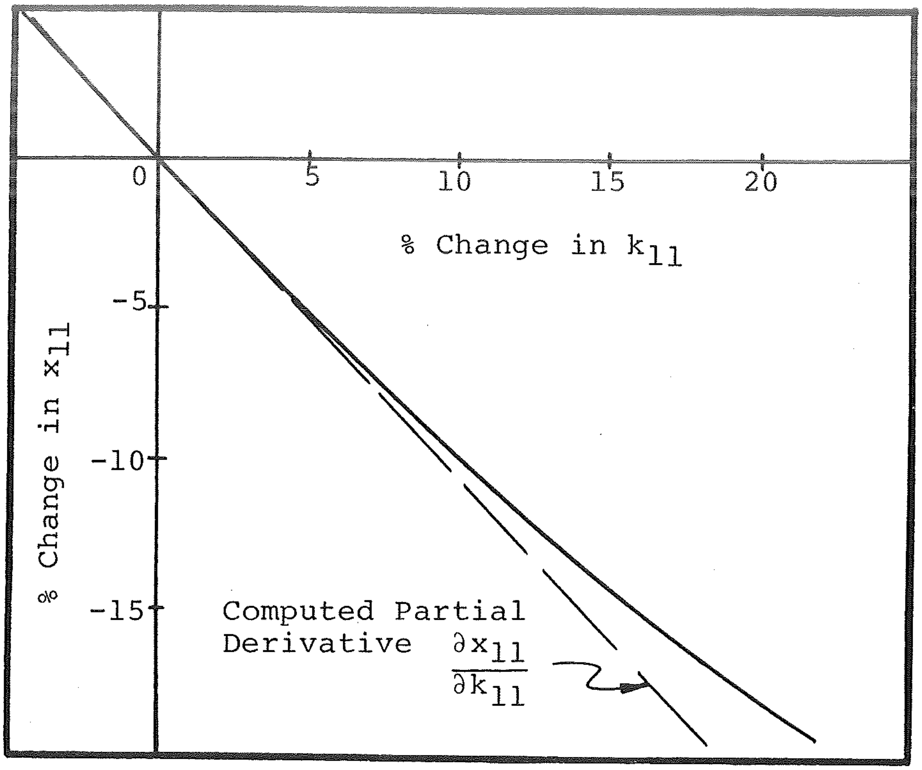


Figure 7-2 The Effect of Changes in k_{11} Upon Changes in x_{11}

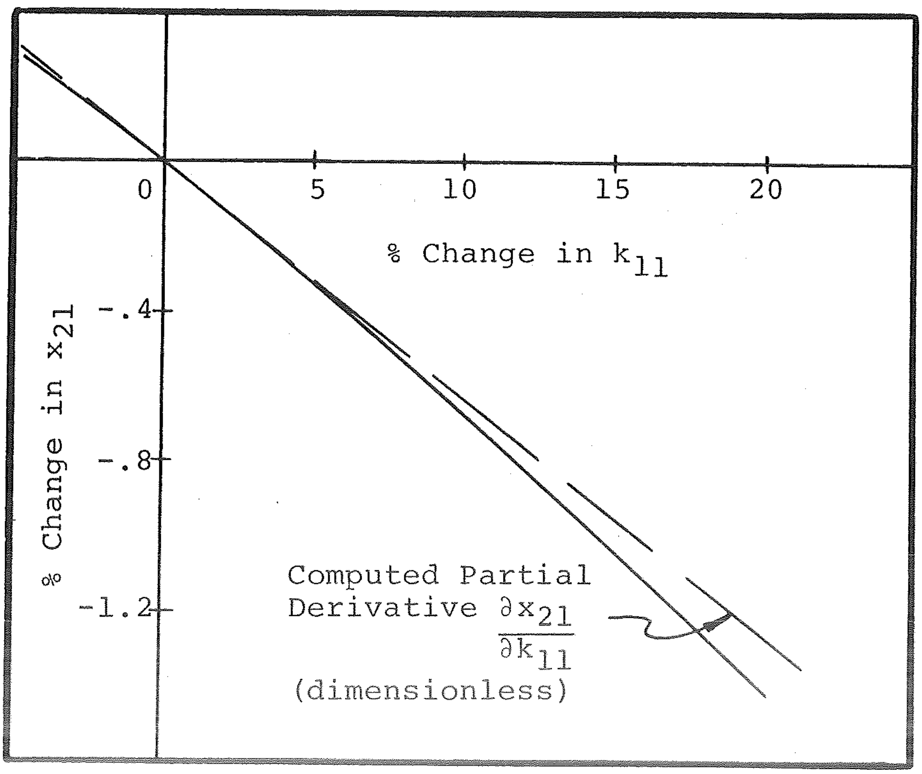


Figure 7-3 The Effect of Changes in k_{11} Upon Changes in x_{21}

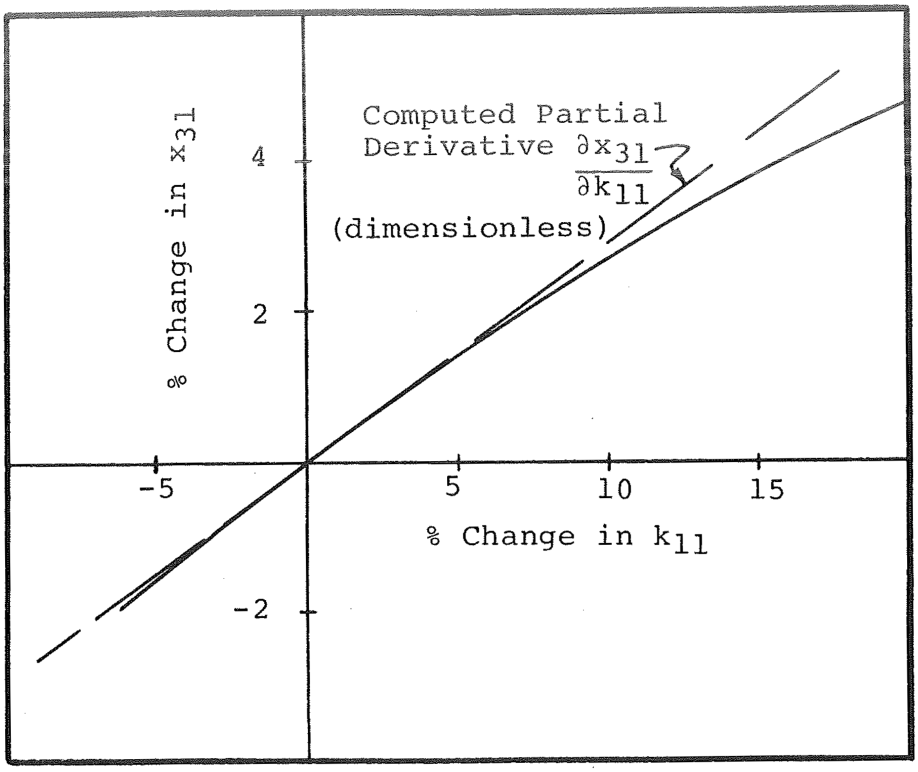


Figure 7-4 The Effect of Changes in k_{11} Upon Changes in x_{31}

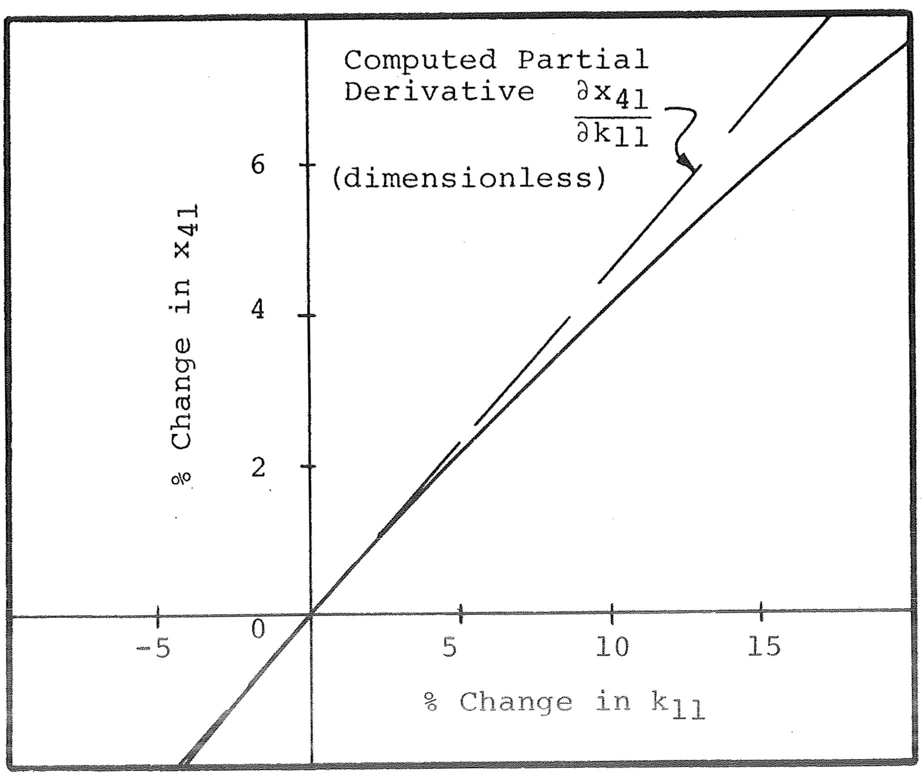


Figure 7-5 The Effect of Changes in k_{11} Upon Changes in x_{41}

7.3 S II Longitudinal Vibration

7.3.1 The Problem and VIDAP Solution

The mass and stiffness matrix data for the 26 degree-of-freedom S II Longitudinal Vibration Analysis are presented in Table 7-3. In this particular problem a section of the stiffness matrix is considered uncertain. This section is located in the upper left-hand corner and the relationships between the stiffnesses are shown more specifically by the matrix below. k_1 and k_2 are considered to be random variables.

$$[K] = \left[\begin{array}{ccc|ccc} k_1 & -k_1 & 0 & \cdot & \cdot & \cdot \\ -k_1 & k_1+k_2+k_3 & -k_2 & \cdot & \cdot & \cdot \\ 0 & -k_2 & k_2+k_4 & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right]$$

The VIDAP program can take any combination of rows and columns in the stiffness and provide eigenvalue/vector statistics based on the property statistics. In this problem the first three rows and columns forming the 3 x 3 section is to be treated by VIDAP. Looking at the stiffness matrix, the following matrix,

$\left[\dfrac{\partial(k_{ij}, m_{ii})}{\partial(k_\ell)}\right]$, can be developed.

TABLE 7-3

S II LONGITUDINAL VIBRATION ANALYSIS REVISION T = 340 SEC

MASS MATRIX

COLUMN(1)	COLUMN(2)	COLUMN(3)	COLUMN(4)	COLUMN(5)	COLUMN(6)	COLUMN(7)	COLUMN(8)	COLUMN(9)	COLUMN(10)
1.0640+01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	1.0098+02	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	2.0700+00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.7890+01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	4.3950+01	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	2.7270+01	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.4091+02	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.9290+01	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.3000+01	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	5.2400+00
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 7-3 (Cont'd)

COLUMN(11)	COLUMN(12)	COLUMN(13)	COLUMN(14)	COLUMN(15)	COLUMN(16)	COLUMN(17)	COLUMN(18)	COLUMN(19)	COLUMN(20)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.8500+01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	2.9268+01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	5.5130+00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	2.5380+01	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	3.0900+00	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	9.2400+00	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	2.3300+01	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.4050+01	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.6950+01	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	2.2180+01
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 7-3 (Cont'd)

COLUMN(21)	COLUMN(22)	COLUMN(23)	COLUMN(24)	COLUMN(25)	COLUMN(26)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9.6000+00	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	1.9650+01	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.4210+01	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	4.9426+02	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.1500+02	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.7400+01

TABLE 7-3 (Cont'd)

STIFFNESS MATRIX

COLUMN(1)	COLUMN(2)	COLUMN(3)	COLUMN(4)	COLUMN(5)	COLUMN(6)	COLUMN(7)	COLUMN(8)	COLUMN(9)	COLUMN(10)
1.4390+05	-1.4390+05	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.4390+05	2.1203+06	-1.6950+06	-2.8140+05	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	-1.6950+06	3.6375+07	-3.4680+07	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	-2.8140+05	-3.4680+07	4.0432+07	-4.2610+06	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	-4.2610+06	7.1660+06	-2.9050+06	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	-2.9050+06	1.7665+07	-1.4760+07	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	-1.4760+07	2.5005+07	-4.4760+06	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-4.4760+06	6.4996+06	-1.0703+06	-7.2240+05
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-1.0703+06	1.9020+06	-7.0375+05
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-7.2240+05	-7.0375+05	2.1818+06
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-4.2230+04
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-3.5790+04
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-4.4240+04
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-9.2110+04
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-2.3090+05	-1.2808+05	-5.4150+05
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-5.7690+06	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	-1.2100+06	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 7-3 (Cont'd)

COLUMN(11)	COLUMN(12)	COLUMN(13)	COLUMN(14)	COLUMN(15)	COLUMN(16)	COLUMN(17)	COLUMN(18)	COLUMN(19)	COLUMN(20)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	-2.3090+05	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	-1.2808+05	0.0000	0.0000	0.0000	0.0000	0.0000
-4.2230+04	-3.5790+04	-4.4240+04	-9.2110+04	-5.4150+05	0.0000	0.0000	0.0000	0.0000	0.0000
1.9145+05	-1.0210+04	-1.1444+05	-2.4570+04	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0210+04	2.0739+05	-1.0552+05	-5.5870+04	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.1444+05	-1.0552+05	3.3640+05	-7.2160+04	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.4570+04	-5.5870+04	-7.2160+04	2.4471+05	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	5.5978+06	-2.6460+06	-5.7850+05	-3.0170+05	-6.0740+05	-3.3240+05
0.0000	0.0000	0.0000	0.0000	-2.6460+06	5.1637+06	-7.1440+04	-7.3990+04	-8.1350+04	-8.6800+04
0.0000	0.0000	0.0000	0.0000	-5.7850+05	-7.1440+04	7.1974+05	-2.6080+04	-1.2500+04	0.0000
0.0000	0.0000	0.0000	0.0000	-3.0170+05	-7.3990+04	-2.6080+04	4.9903+05	-6.2890+04	0.0000
0.0000	0.0000	0.0000	0.0000	-6.0740+05	-8.1350+04	-1.2500+04	-6.2890+04	8.5862+05	-5.4590+04
0.0000	0.0000	0.0000	0.0000	-3.3240+05	-8.6800+04	0.0000	0.0000	-5.4590+04	5.0768+05
0.0000	0.0000	0.0000	0.0000	-2.3133+05	-2.1306+06	-1.6490+04	-1.6240+04	-1.7960+04	-1.5600+04
0.0000	0.0000	0.0000	0.0000	0.0000	-7.3540+04	-1.4730+04	-1.8130+04	-2.1930+04	-1.8290+04
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 7-3 (Concl.)

COLUMN(21)	COLUMN(22)	COLUMN(23)	COLUMN(24)	COLUMN(25)	COLUMN(26)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	-1.2100+06	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	-5.7690+06	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.3133+05	0.0000	0.0000	0.0000	0.0000	0.0000
-2.1306+06	-7.3540+04	0.0000	0.0000	0.0000	0.0000
-1.6490+04	-1.4730+04	0.0000	0.0000	0.0000	0.0000
-1.6240+04	-1.8130+04	0.0000	0.0000	0.0000	0.0000
-1.7960+04	-2.1930+04	0.0000	0.0000	0.0000	0.0000
-1.5600+04	-1.8290+04	0.0000	0.0000	0.0000	0.0000
3.2698+06	-8.4157+05	0.0000	0.0000	0.0000	0.0000
-8.4157+05	2.8662+06	-1.8780+06	0.0000	0.0000	0.0000
0.0000	-1.8780+06	1.8780+06	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	5.7690+06	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.3780+06	-1.6800+05
0.0000	0.0000	0.0000	0.0000	-1.6800+05	1.6800+05

$$
\begin{Bmatrix} dk_{11} \\ dk_{12} \\ dk_{13} \\ dk_{22} \\ dk_{23} \\ dk_{33} \\ dm_{11} \\ dm_{22} \\ dm_{33} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} dk_1 \\ dk_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

or

$$
\{ dk_{ij},\ dm_{ii} \} = \left[\frac{\partial (k_{ij}, m_{ii})}{\partial (k_\ell)} \right] \{ dk_\ell \}
$$

This procedure is exactly the same as that discussed in the four degree-of-freedom system discussed in the previous section. $\left[\frac{\partial (k_{ij},\ m_{ii})}{\partial (k_\ell)} \right]$ is equivalent to the A (I,J) matrix discribed in Appendix B (Step 18) and must have nine columns.

The stiffnesses k_1 and k_2 are assumed to have the following mean values and standard deviations.

$$\overline{k_1} = 143{,}900 \text{ in-lb.}$$

$$\sigma_{k_1} = .10k_1 = 14{,}390 \text{ in-lb.}$$

$$\overline{k_2} = 1{,}695{,}000 \text{ in-lb.}$$

$$\sigma_{k_2} = .10k_2 = 169{,}500 \text{ in-lb.}$$

k_1 and k_2 are also assumed to be statistically independent. The 9 x 9 covariance matrix is as follows:

$$
[\Sigma] = \underbrace{\begin{bmatrix} \sigma_{k_1}^2 & 0 & 0 & 0 & \cdot & 0 \\ 0 & \sigma_{k_2}^2 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 \end{bmatrix}}_{\text{9 columns}}
$$

The [KR] and [KS] matrix give the row indices and column indices respectively of the rows and columns of the stiffness matrix under consideration. Thus

$$
[KR] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}
$$

and $[KS] = [KR]'$

Statistical data were requested for eigenvalues and eigenvectors 2-12 (the first eigenvalue is zero, a rigid body mode).

The statistics of the first four (1-4) and the last four (23-26) elements of these eigenvectors were requested for computation and printout. The statistical results of the VIDAP run are presented in Tables 7-4, 5, and 6.

Several observations can be made from these results which are significant in the analysis of the statistical characteristics of a system.

(1) The modes which have the greatest uncertainty in the eigenvalue or frequency also have the greatest uncertainty in the eigenvector components. This is substantiated also by an examination of the partial derivatives. Note in Equation (3-30) that the partial derivative of an eigenvector component with respect to a stiffness element contains the expression for the partial derivative of the eigenvalue with respect

to the stiffness. The dependency is not one to one, but generally an increase in $\frac{\partial \lambda_i}{\partial k_{rs}}$ will lead to an increase in $\left\{ \frac{\partial x_i}{\partial k_{rs}} \right\}$.

(2) Large statistical correlations exist between all combinations of eigenvalues and eigenvector components. This indicates that in using the statistical characteristics of the modes and frequencies one must consider covariance.

(3) Percentage variations in modal characteristics are generally smaller than the percentage variations in the stiffnesses. This is not a fixed rule because the uncertainty in the system will grow as more stiffness elements are allowed to become random. However previous observations of various dynamic systems along with the two in this report generally support the trend of modal uncertainty being smaller than stiffness uncertainty.

TABLE 7-4 Eigenvalue and Frequency Statistics of the S II Longitudinal Vibration Model (VIDAP)

MODE	EIGENVALUE	STANDARD DEVIATION	FREQUENCY	STANDARD DEVIATION
2	2.17902900E+03	7.47116776E-01	7.42936412E+00	1.27364132E-03
3	3.56577760E+03	3.60208081E+00	9.50379949E+00	4.80027902E-03
4	4.81187550E+03	8.95562946E+00	1.10402099E+01	1.02737521E-02
5	8.87489820E+03	1.89448827E+02	1.49934587E+01	1.60029625E-01
6	9.19593230E+03	6.54030261E-02	1.52622310E+01	5.42737859E-05
7	1.11712580E+04	3.37852999E-04	1.68217502E+01	2.54378118E-07
8	1.31353400E+04	5.44614053E+02	1.82406705E+01	3.78144982E-01
9	1.81748840E+04	8.13501297E+02	2.14563558E+01	4.80189416E-01
10	2.05058600E+04	1.27598538E-01	2.27907771E+01	7.09082654E-05
11	2.43924000E+04	2.68795868E-02	2.48569252E+01	1.36957392E-05
12	2.82534640E+04	5.07916592E-01	2.67519836E+01	2.87804979E-04

TABLE 7-5 Eigenvector Statistics of the S II Longitudinal Vibration Model(VIDAP)

VECTOR	EIGENVECTOR	STANDARD DEVIATION
X(2, 1)	1.99119650E-02	4.51323386E-05
X(2, 2)	1.67037910E-02	1.49444896E-04
X(2, 3)	1.46252600E-02	4.37620047E-05
X(2, 4)	1.45217680E-02	4.19744743E-05
X(2, 23)	-2.00814270E-02	3.59035882E-05
X(2, 24)	7.84932150E-03	2.64354296E-05
X(2, 25)	1.92997130E-02	2.46352988E-05
X(2, 26)	2.49248910E-02	2.65433067E-05
X(3, 1)	2.98029150E-02	9.32199678E-04
X(3, 2)	2.19452440E-02	4.45513361E-04
X(3, 3)	1.74071270E-02	6.75449781E-04
X(3, 4)	1.71816200E-02	6.66971408E-04
X(3, 23)	-7.11829650E-02	4.79887636E-04
X(3, 24)	-2.12488950E-03	1.79054532E-04
X(3, 25)	2.96336370E-02	1.19700142E-04
X(3, 26)	4.69862580E-02	1.44151076E-04
X(4, 1)	3.87601210E-02	2.35031101E-04
X(4, 2)	2.49696110E-02	1.36543221E-03
X(4, 3)	1.78767800E-02	5.41642603E-04
X(4, 4)	1.75249800E-02	5.36054192E-04
X(4, 23)	3.87443620E-02	4.40657942E-04
X(4, 24)	-2.62685090E-02	2.71161228E-04
X(4, 25)	4.33003400E-02	2.35774197E-04
X(4, 26)	8.63197640E-02	1.80140359E-04
X(5, 1)	1.42735490E-01	2.08507293E-02
X(5, 2)	4.90707370E-02	1.65646990E-02
X(5, 3)	2.02026560E-02	9.84993443E-03
X(5, 4)	1.87810140E-02	9.57504289E-03
X(5, 23)	1.43928740E-03	7.80050527E-03
X(5, 24)	-7.40163090E-03	4.14842458E-03
X(5, 25)	-1.32010330E-02	3.36758459E-03
X(5, 26)	-1.63350600E-01	2.89743809E-03
X(6, 1)	2.63405500E-03	1.24071891E-04
X(6, 2)	8.43030870E-04	3.55212386E-04
X(6, 3)	3.20195930E-04	1.60837705E-04
X(6, 4)	2.94466390E-04	1.53613335E-04
X(6, 23)	-3.65282150E-03	1.07470930E-04
X(6, 24)	-8.74285120E-04	2.92724795E-05
X(6, 25)	-1.10942780E-04	1.94043554E-05
X(6, 26)	-2.33248760E-03	1.67080258E-05

TABLE 7-5 (Cont'd)

X(7, 1)	1.77708750E-04	2.25349550E-06
X(7, 2)	3.09203500E-05	6.26096913E-05
X(7, 3)	2.78043030E-06	3.40022325E-05
X(7, 4)	1.40322560E-06	3.24780403E-05
X(7, 23)	-3.42423150E-03	2.23692801E-05
X(7, 24)	-7.56410330E-04	3.76190536E-06
X(7, 25)	1.45966880E-06	8.02180267E-08
X(7, 26)	-9.29588140E-06	9.82879623E-08
X(8, 1)	1.99809180E-01	5.95195507E-03
X(8, 2)	5.74847880E-03	7.90019566E-03
X(8, 3)	-1.21158860E-02	3.38517062E-03
X(8, 4)	-1.29795150E-02	3.14906080E-03
X(8, 23)	2.45007720E-04	2.56485013E-03
X(8, 24)	3.47635650E-03	1.24260058E-03
X(8, 25)	-4.70085400E-02	9.54851531E-04
X(8, 26)	1.30639670E-01	8.61961569E-04
X(9, 1)	1.72781740E-01	1.84648566E-02
X(9, 2)	-5.94119040E-02	2.25320420E-03
X(9, 3)	-2.14151110E-02	2.70118938E-03
X(9, 4)	-1.95347700E-02	2.70599884E-03
X(9, 23)	1.89128640E-03	2.41301357E-03
X(9, 24)	4.30014770E-03	1.28423464E-03
X(9, 25)	4.53059150E-02	1.04013944E-03
X(9, 26)	-5.13440040E-02	9.96582212E-04
X(10, 1)	1.67803140E-03	2.36745705E-05
X(10, 2)	-8.66213450E-04	2.62196858E-05
X(10, 3)	-1.48940040E-04	6.35678035E-05
X(10, 4)	-1.13700690E-04	6.21099072E-05
X(10, 23)	-1.48189970E-01	5.27091359E-05
X(10, 24)	-7.88292670E-04	2.37436169E-05
X(10, 25)	1.65619960E-04	1.77875528E-05
X(10, 26)	-1.47372160E-04	1.73078416E-05
X(11, 1)	4.46285540E-04	7.32132695E-06
X(11, 2)	-3.58626050E-04	6.37497282E-06
X(11, 3)	2.70387150E-05	3.22479882E-05
X(11, 4)	4.58488800E-05	3.15649402E-05
X(11, 23)	-9.90264940E-03	2.73705224E-05
X(11, 24)	-4.08363340E-04	1.10371050E-05
X(11, 25)	-4.21219610E-05	7.37019219E-06
X(11, 26)	2.75964250E-05	7.44144042E-06
X(12, 1)	1.33145450E-03	3.16768811E-05
X(12, 2)	-1.45004420E-03	1.28259106E-05
X(12, 3)	4.27686250E-04	1.61995737E-04
X(12, 4)	5.18739880E-04	1.60760684E-04
X(12, 23)	-4.09504890E-02	1.47990669E-04
X(12, 24)	-1.55037420E-03	6.60967292E-05
X(12, 25)	-3.51849250E-04	4.65768821E-05
X(12, 26)	1.82660060E-04	4.80995879E-05

TABLE 7-6 Correlation Matrix for S II

Longitudinal Vibrations*

λ_2, ω_2					
1.00000000E+00					
x(2,1)					
9.60551684E-01	1.00000000E+00				
x(2,2)					
-8.69723113E-01	-9.72668224E-01	1.00000000E+00			
x(2,3)					
9.49522652E-01	8.24825555E-01	-6.70999058E-01	1.00000000E+00		
x(2,4)					
9.52321666E-01	8.29906571E-01	-6.77679192E-01	9.99959086E-01	1.00000000E+00	
x(2,23)					
9.74929086E-01	8.74587792E-01	-7.38097922E-01	9.95520116E-01	9.96334724E-01	1.00000000E+00
x(2,24)					
9.99688215E-01	9.53308162E-01	-8.57128536E-01	9.57059476E-01	9.59642814E-01	9.80181210E-01
1.00000000E+00					
x(2,25)					
9.99492123E-01	9.68926064E-01	-8.85008970E-01	9.39043825E-01	9.42115558E-01	9.67343062E-01
9.98334799E-01	1.00000000E+00				
x(2,26)					
9.99987745E-01	9.61916746E-01	-8.72135888E-01	9.47957946E-01	9.50799517E-01	9.73815500E-01
9.99552344E-01	9.99637641E-01	1.00000000E+00			
λ_3, ω_3					
9.98766661E-01	9.45559162E-01	-8.44146021E-01	9.63926805E-01	9.66295241E-01	9.84774653E-01
9.99695002E-01	9.96677210E-01	9.98508610E-01	1.00000000E+00		
x(3,1)					
9.92133399E-01	9.87809585E-01	-9.24665219E-01	9.02782748E-01	9.05636671E-01	9.39404037E-01
9.88698265E-01	9.95618774E-01	9.92741010E-01	9.84694277E-01	1.00000000E+00	
x(3,2)					
9.48472927E-01	8.22938520E-01	-6.68526647E-01	9.99994457E-01	9.99923417E-01	9.95199792E-01
9.56088974E-01	9.37893943E-01	9.46892591E-01	9.63035255E-01	9.01345732E-01	1.00000000E+00
x(3,3)					
9.99998582E-01	9.60081917E-01	-8.68890614E-01	9.50049665E-01	9.52834186E-01	9.75302485E-01
9.99728853E-01	9.99437032E-01	9.99977988E-01	9.98848870E-01	9.91921143E-01	9.49005263E-01
1.00000000E+00					
x(3,4)					
9.99998077E-01	9.61095183E-01	-8.70689252E-01	9.48905676E-01	9.51721554E-01	9.74490867E-01
9.99637329E-01	9.99552691E-01	9.99995530E-01	9.98657379E-01	9.92376974E-01	9.47849757E-01
9.99993356E-01	1.00000000E+00				

*This contains only the first 14 rows of the VIDAP output. The complete matrix is symmetric and has 99 rows and columns.

7.3.2 Checkout of VIDAP Partial Derivatives

A number of perturbation runs were made on the S II Longitudinal Vibration model. In each case k_{11}, k_{12}, k_{21}, and k_{22} in various combinations were perturbed and partial derivatives were numerically developed to compare with those computed by VIDAP. Several typical partial derivatives are plotted in Figures 7-6 through 7-15. The partial derivative checks for eigenvalues were excellent, but those for eigenvector components were not altogether satisfactory. It appears that numerical roundoff of the eigenvectors coupled with peculiarities of the problem caused the perturbed model to produce partial derivatives which frequently varied significantly from the VIDAP computed partial derivatives. Basic observations are:

(1) The larger partial derivatives (those which are of the greatest significance) can be computed the most accurately. The smaller partials wander and may even have reversals of sign.

(2) Eigenvectors which have only a few dominant components, such as mode 9, produce reasonably accurate partial derivatives for those components.

The S II Longitudinal Vibrational Model appeared to have some peculiarities when perturbed which were difficult to explain. Occasionally adjoining eigenvectors would alter characteristics significantly when the roots became close. The partial derivatives of the first several eigenvectors did not agree at all with the computed partial derivatives. However the small value of these partial derivatives (regardless of accuracy) resulted in small uncertainties for the components and the resulting statistics are of the same small magnitude shown in the Monte Carlo check to be discussed in the next section.

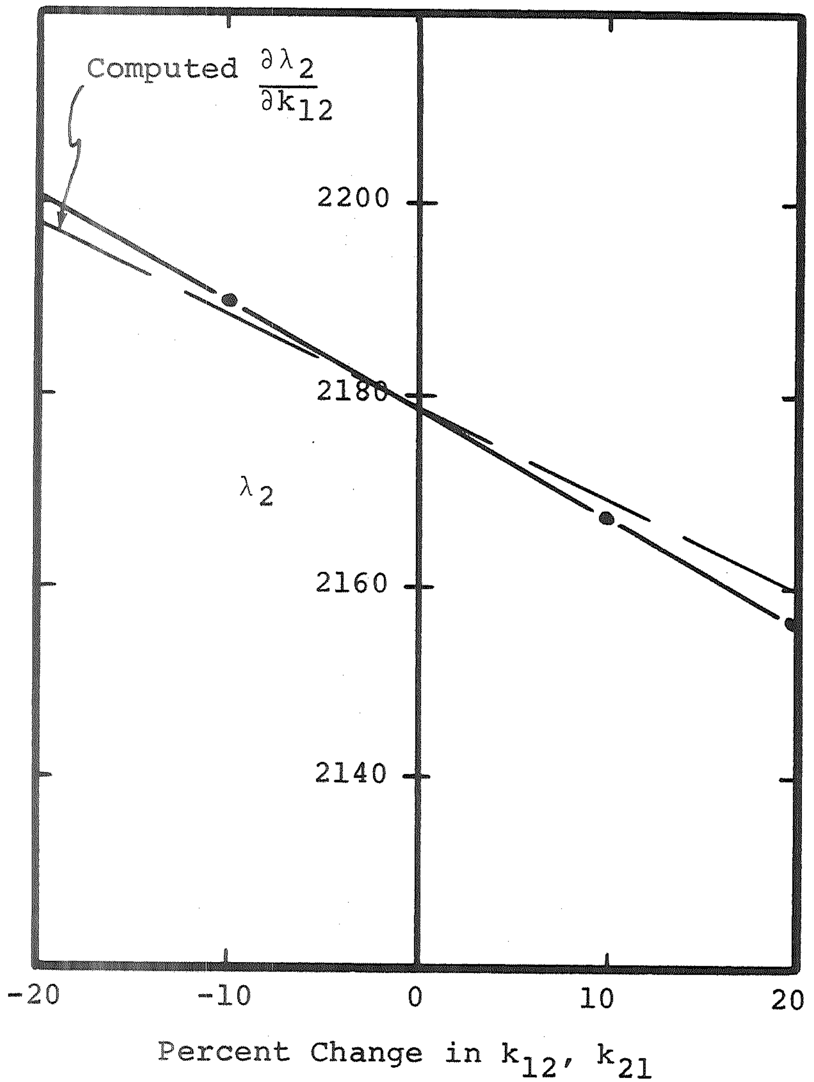


Figure 7-6 Partial Derivative Check, $\frac{\partial \lambda_2}{\partial k_{12}}$

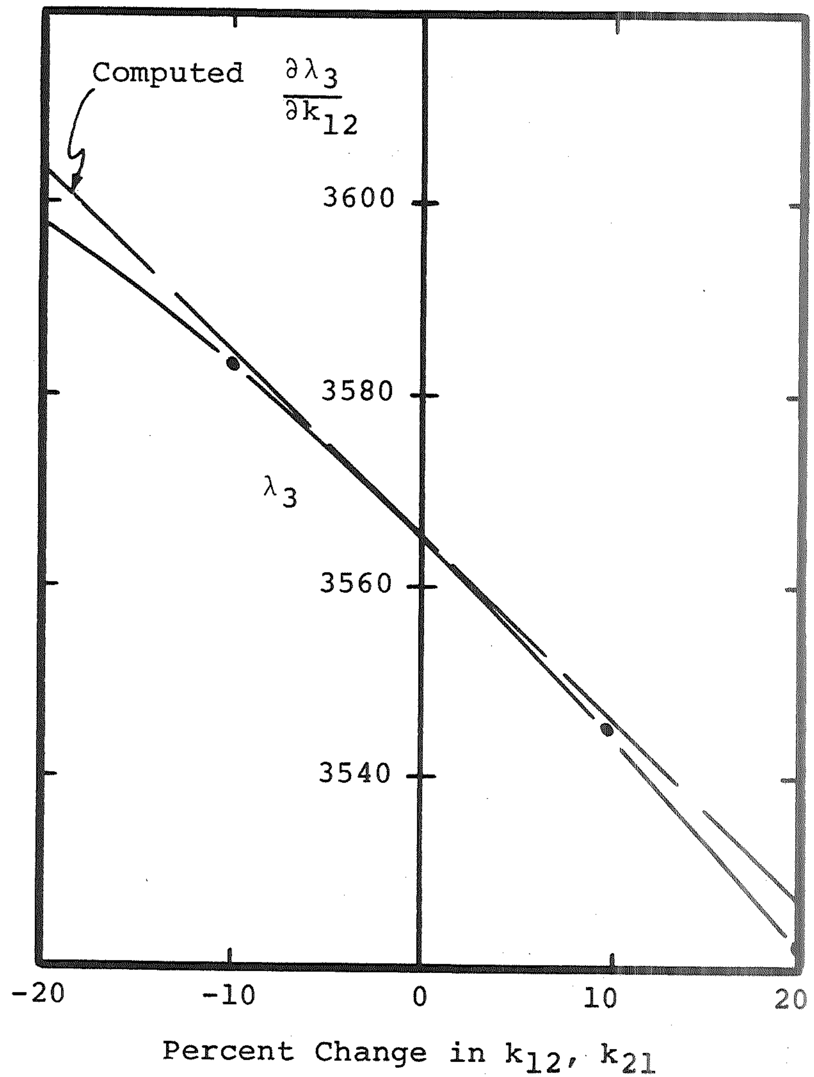


Figure 7-7 Partial Derivative Check, $\frac{\partial \lambda_3}{\partial k_{12}}$

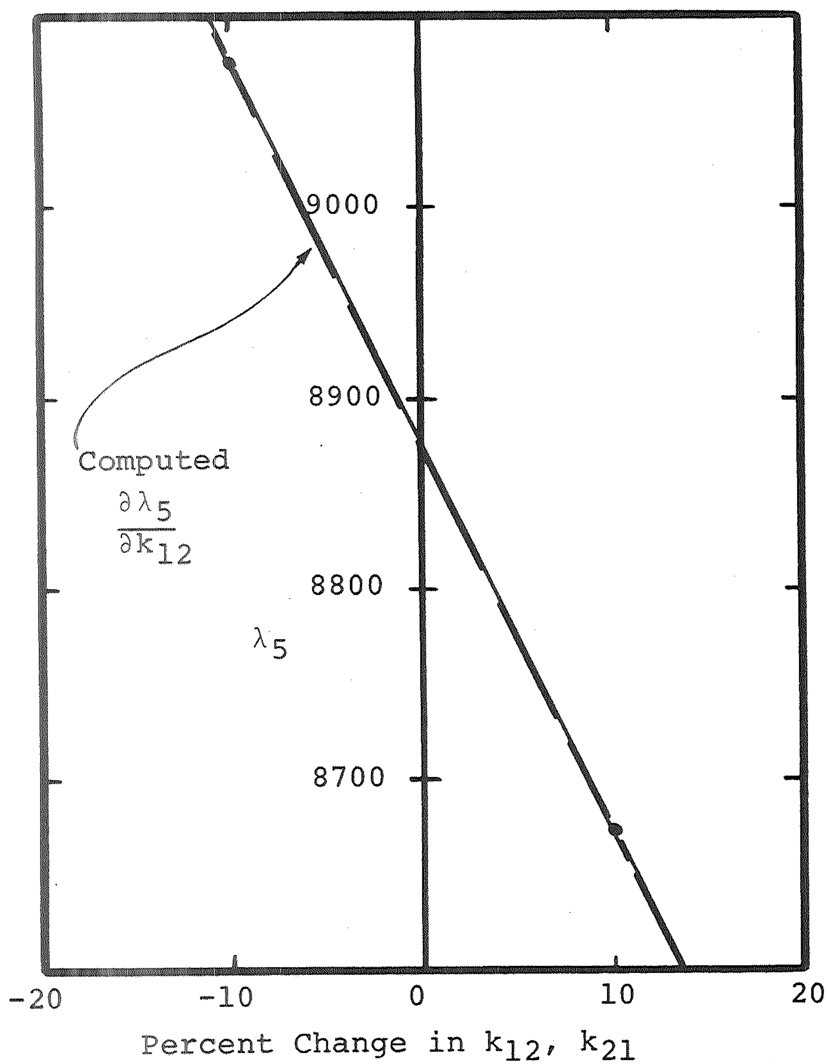


Figure 7-8 Partial Derivative Check, $\frac{\partial \lambda_5}{\partial k_{12}}$

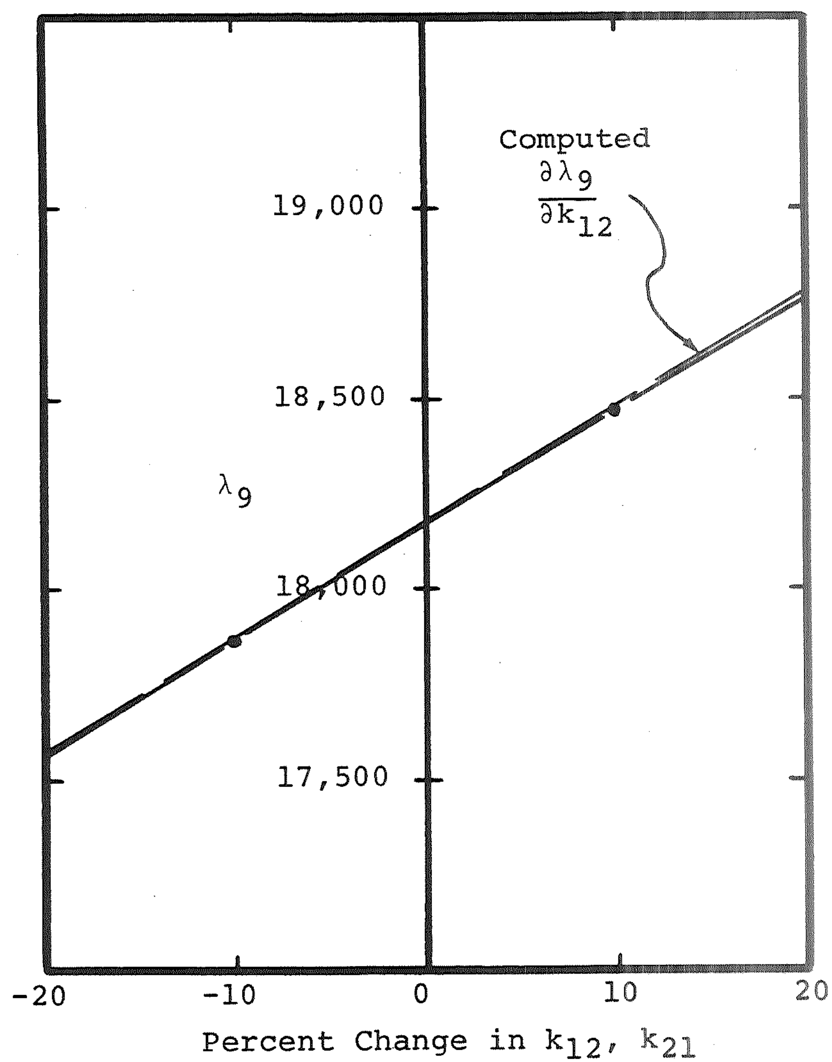


Figure 7-9 Partial Derivative Check, $\frac{\partial \lambda_9}{\partial k_{12}}$

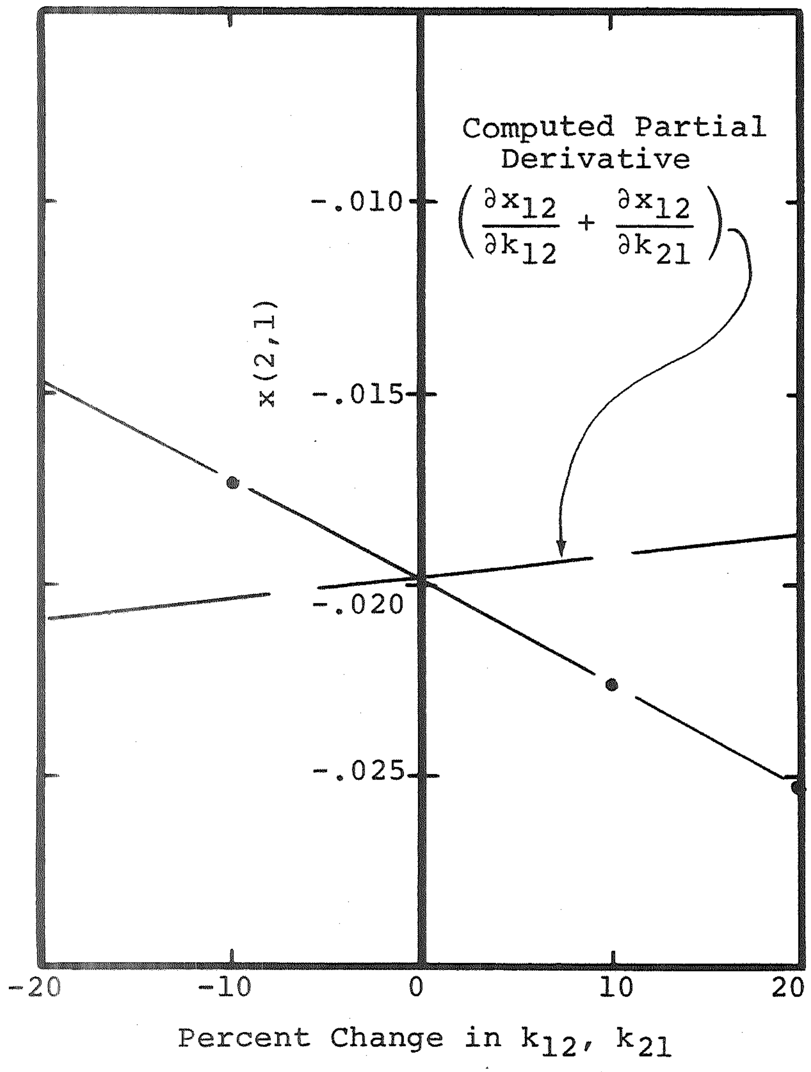


Figure 7-10 Partial Derivative Check, $\frac{\partial x_{12}}{\partial k_{12}}$

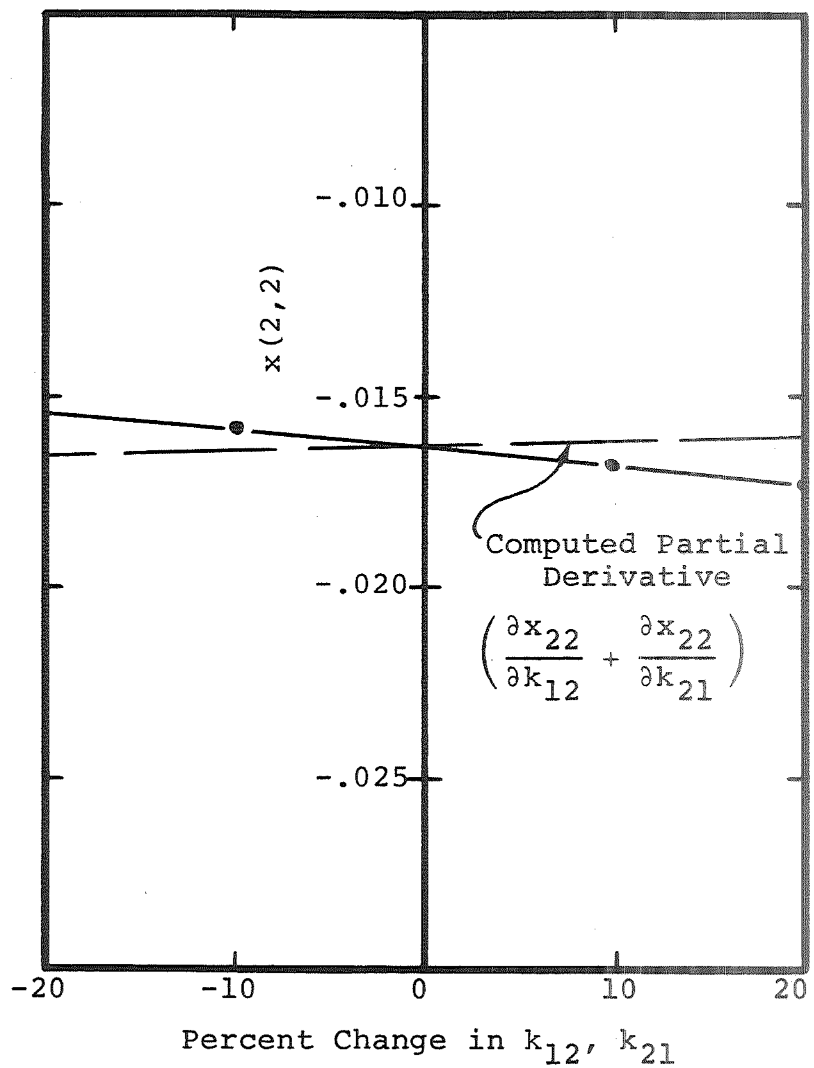


Figure 7-11 Partial Derivative Check, $\frac{\partial x_{22}}{\partial k_{12}}$

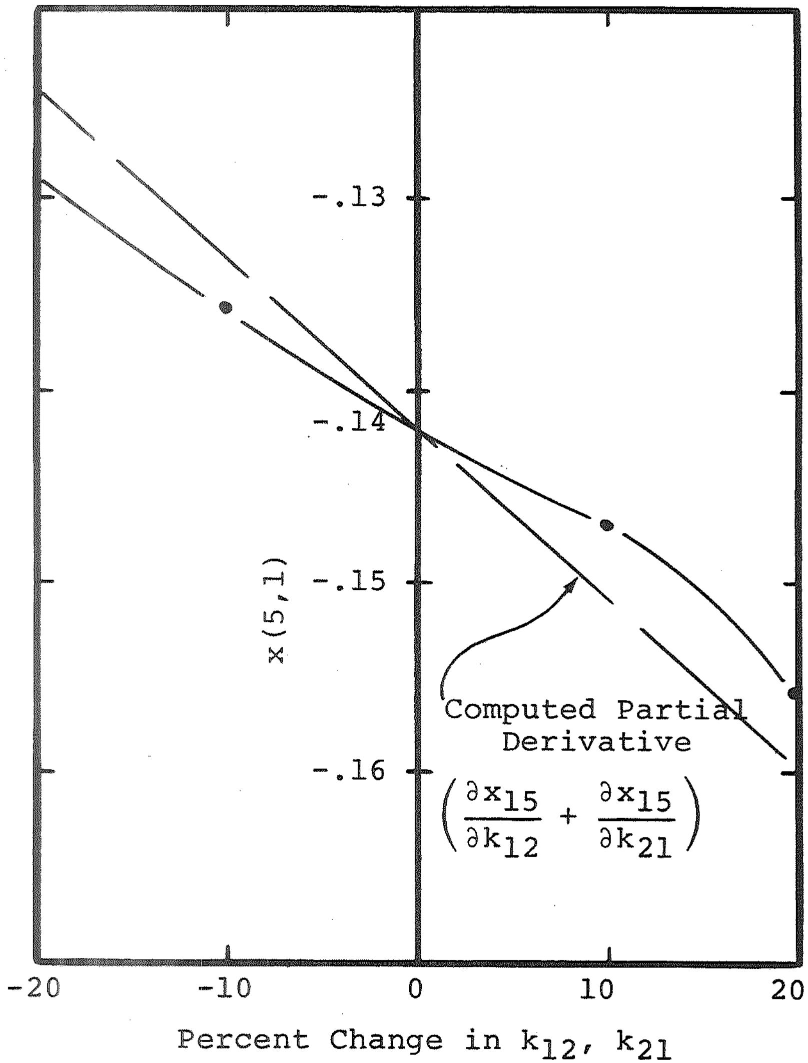


Figure 7-12 Partial Derivative Check, $\frac{\partial x_{15}}{\partial k_{12}}$

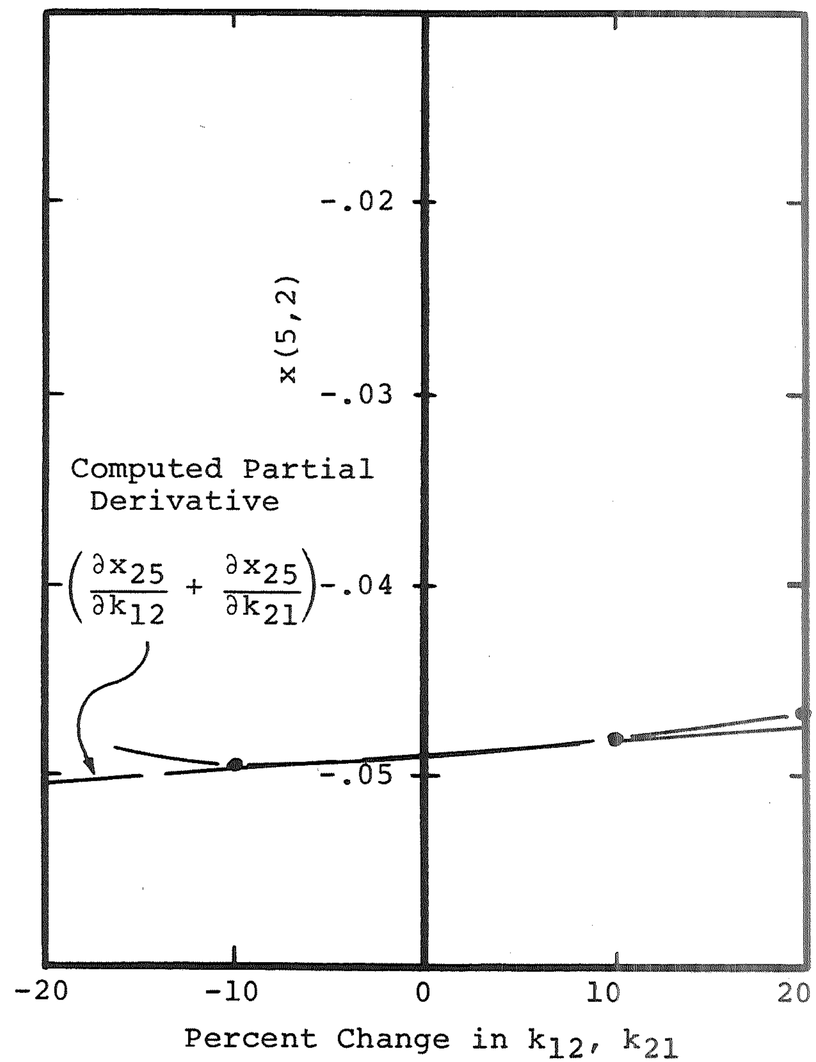


Figure 7-13 Partial Derivative Check, $\frac{\partial x_{25}}{\partial k_{12}}$

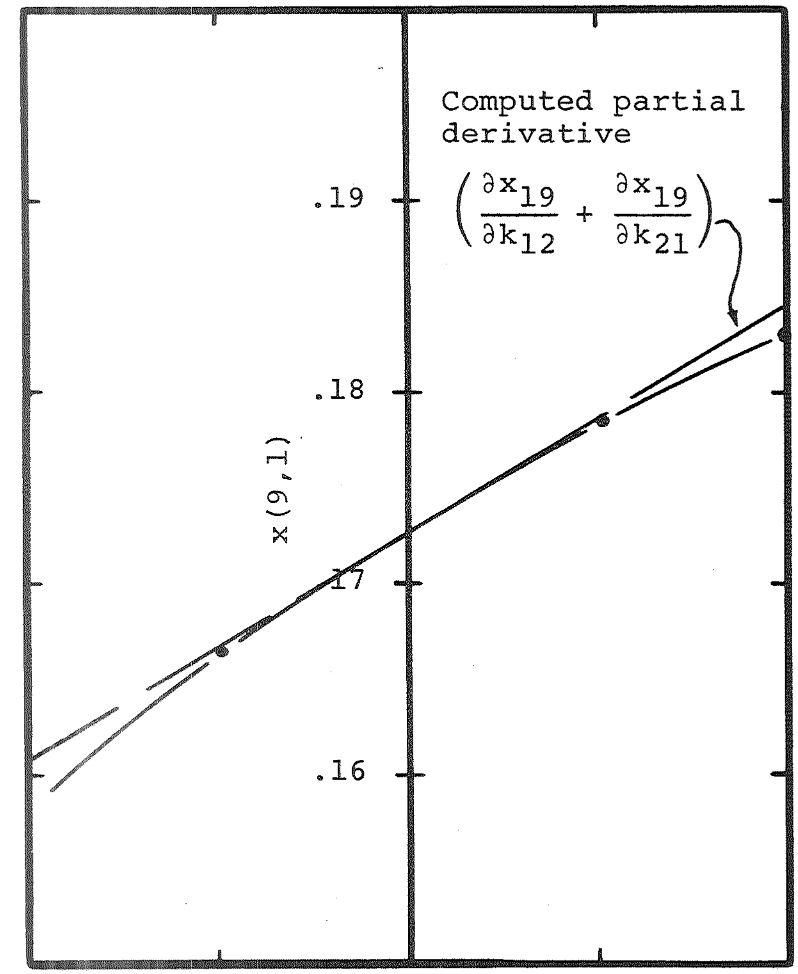


Figure 7-14 Partial Derivative Check, $\frac{\partial x_{19}}{\partial k_{12}}$

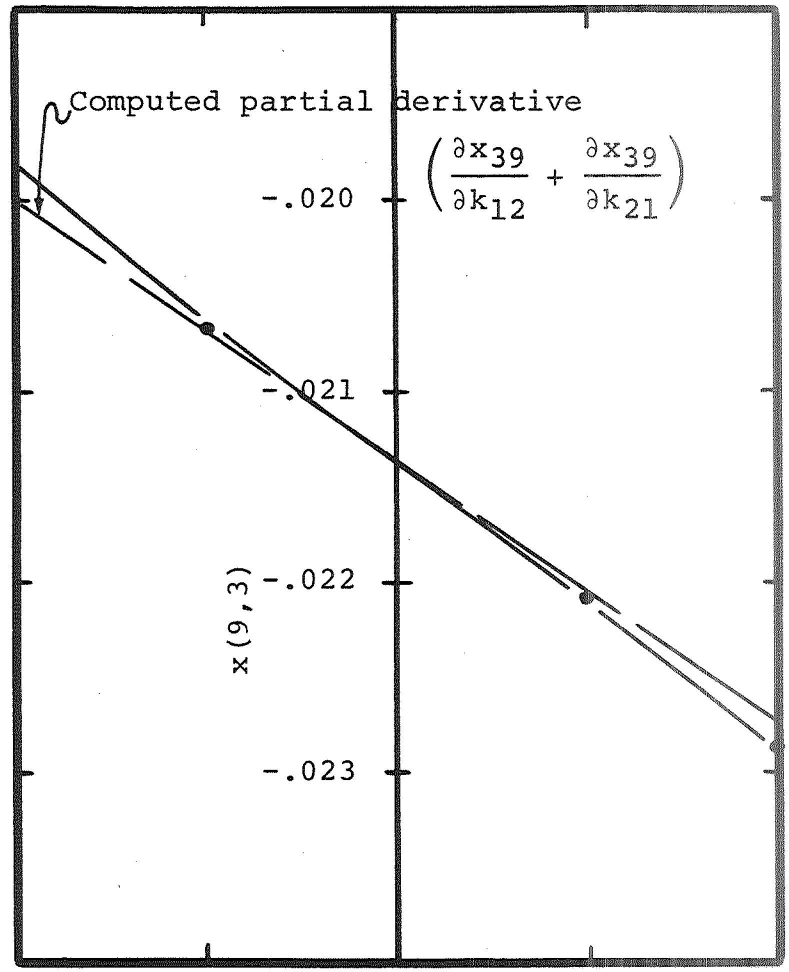


Figure 7-15 Partial Derivative Check, $\frac{\partial x_{39}}{\partial k_{12}}$

7.3.3 Monte Carlo Check

The purpose of this Monte Carlo analysis is to offer a completely independent statistical check of the linear statistical model. The procedure used in this analysis is described in the flow diagram in Figure 7-16. In the analysis the procedure was repeated 30 times. Both the estimated mean and standard deviation of a Monte Carlo analysis converge at the rate of $1/\sqrt{N}$. Hence a 60-run analysis would have increased the accuracy by $\sqrt{2}$.

Any statistical analysis using a finite sample only results in estimates of true means and variances. Hence confidence limits or bounds are included to establish the ranges within which the true means and standard deviations are most likely to fall.

The stiffness matrix with its dependency upon k_1 and k_2 is shown at the beginning of Section 7.3.1. Using the procedure described in Figure 7-16, the following matrix relation was developed to relate stiffness matrix elements to the random stiffnesses k_1 and k_2.

$$\begin{Bmatrix} k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \\ k_{23} \\ k_{32} \\ k_{33} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \overline{k_1} + z_1\sigma_{k_1} \\ \overline{k_2} + z_2\sigma_{k_2} \\ k_3 \\ k_4 \end{Bmatrix}$$

$$= \left[\frac{\partial(k_{11}, k_{12}, \ldots)}{\partial(k_1, k_2, \ldots)} \right] \begin{Bmatrix} \overline{k_1} + z_1\sigma_{k_1} \\ \overline{k_2} + z_2\sigma_{k_2} \\ k_3 \\ k_4 \end{Bmatrix}$$

where z_1 and z_2 are independent normally distributed random variables with zero means and standard deviations equal to one.

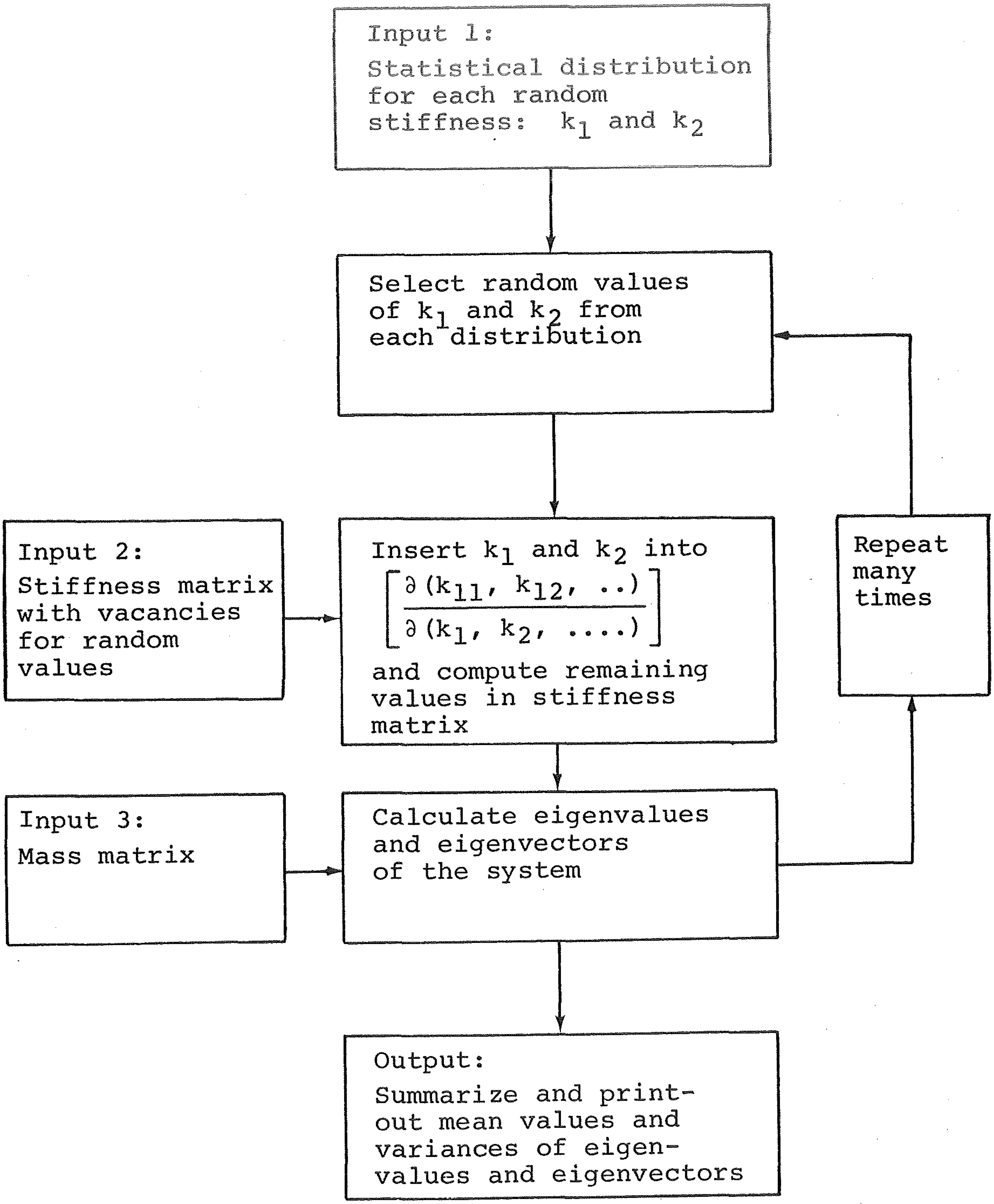


Figure 7-16 Summary of Monte Carlo Procedure

The results of the check are summarized in Tables 7-7 and 7-8. Note that the frequency table shows very excellent correlation between the VIDAP linear statistical model and the Monte Carlo. This should be expected since the eigenvalues are relatively well-behaved and the partial derivatives for eigenvalues are simple and are not as easily influenced by accumulative round-off.

Although Monte Carlo results are not shown for eigenvector components of modes 2, 3, 4, 6, 10, 11, and 12 it was confirmed that these modes, which have small deviations in the frequencies, also have small deviations in the eigenvector components.

The mean values in Table 7-8 show excellent agreement, but the standard deviations show only fair agreement. However, except in the case of x(5,2), the standard deviations predicted by VIDAP are of the same magnitude. The lack of finer accuracy can be due to the following factors:

(1) nonlinearity in the eigenvector derivatives
(2) roundoff error
(3) possible unusual behavior of the eigenvectors
(4) the limited size of the Monte Carlo.

This project ended before the above areas could be investigated, but the influences of these effects should be thoroughly evaluated before VIDAP is put into general operation.

TABLE 7-7 A Comparison of VIDAP and Monte Carlo Computed Statistics of Frequencies (S II Longitudinal Vibration)

Mode	Mean			Standard Deviation			
	VIDAP	Monte Carlo	95% ±Conf. Int.	VIDAP	Monte Carlo	95% Confidence Bounds Lower Bound	95% Confidence Bounds Upper Bound
2	7.43	7.43	.001	.001	.001	.001	.002
3	9.50	9.51	.002	.005	.005	.004	.007
4	11.04	11.04	.004	.010	.010	.009	.015
5	14.99	14.97	.060	.160	.160	.128	.212
6	15.26	15.27	.000	.000	.000	.000	.000
7	16.82	16.83	.001	.000	.002	.002	.003
8	18.24	18.20	.124	.378	.331	.261	.439
9	21.46	21.50	.159	.480	.425	.335	.564
10	22.791	22.82	.001	.000	.002	.001	.002
11	24.86	24.87	.000	.000	.001	.001	.001
12	26.75	26.77	.001	.000	.002	.002	.003

TABLE 7-8 A Comparison of VIDAP and Monte Carlo Computed Statistics of Eigenvector Components (S II Longitudinal Vibration)

Vector Component	Mean			Standard Deviation			
	VIDAP	Monte Carlo	±95% Conf. Int.	VIDAP	Monte Carlo	95% Confidence Bounds Lower B.	Upper B.
x(5,1)	.142	.146	.006	.021	.016	.013	.021
x(5,2)	.049	.049	.001	.017	.002	.002	.003
x(5,3)	.020	.020	.001	.010	.002	.002	.003
x(5,4)	.019	.018	.001	.010	.002	.002	.003
x(5,23)	.001	.001	.000	.008	.000	.000	.000
x(5,24)	-.007	-.007	.000	.004	.001	.001	.001
x(5,25)	-.013	-.014	.001	.003	.003	.002	.004
x(5,26)	-.163	-.162	.002	.003	.006	.005	.008
x(8,1)	.200	.197	.004	.006	.012	.009	.016
x(8,2)	.006	.004	.003	.008	.009	.007	.012
x(8,3)	-.012	-.013	.001	.003	.003	.002	.004
x(8,4)	-.013	-.013	.001	.003	.003	.002	.004
x(8,23)	.000	.000	.000	.003	.000	.000	.000
x(8,24)	.003	.004	.000	.001	.001	.001	.000
x(8,25)	-.047	-.046	.002	.001	.005	.004	.007
x(8,26)	.131	.132	.002	.001	.005	.004	.007
x(9,1)	.173	.170	.010	.018	.027	.021	.036
x(9,2)	-.059	-.059	.001	.002	.002	.002	.003
x(9,3)	-.021	-.021	.001	.003	.002	.002	.003
x(9,4)	-.020	-.019	.001	.003	.002	.002	.003
x(9,23)	.002	.002	.000	.002	.001	.001	.001
x(9,24)	.004	.004	.000	.001	.000	.000	.000
x(9,25)	.005	.005	.002	.001	.005	.004	.007
x(9,26)	-.051	-.052	.004	.001	.010	.008	.011

8.0 CONCLUSIONS AND RECOMMENDATIONS

The following conclusions can be made with regard to the bending vibration data accuracy project and the development of VIDAP:

(1) a linear statistical model can be used to develop eigenvalue/vector statistics from stiffness and mass matrix uncertainties.

(2) a very large system can be treated statistically without matrix size requirements which exceed the size of the system.

(3) the statistical computation is faster than an eigenvalue computation since the most complex operation is an (n-1) simultaneous equation solution which, in turn, is faster than a matrix inversion.

(4) beam and plate elements can be modeled statistically and these characteristics can be used to compute eigenvalue/vector statistical characteristics.

(5) large uncertainties of eigenvalues indicate large uncertainties of eigenvectors. In seeking eigenvectors with large uncertainties, VIDAP can be used twice in sequence: first, to compute only eigenvalue statistics of the system to locate uncertainty, and second, to compute eigenvector statistics of those modes indicated by the first run.

(6) only eigenvectors of modes of interest are needed in the computation.

(7) eigenvector roundoff errors and nonlinearities have a very strong influence on the accuracy of the eigenvector statistics.

The following recommendations are made with regard to future research and development of VIDAP and other structural dynamic statistical models:

(1) the effect of roundoff should be thoroughly evaluated and an indicator of the eigenvalue statistics accuracy should be developed. This indicator could

be included with a subroutine which would iterate the eigenvectors to the degree of accuracy required.

(2) An investigation should be made of the application of the VIDAP partial derivatives to other applications such as optimization and improvement of stiffness matrices by incorporation of test results.

(3) Second derivatives of the eigenvalues should be developed to use as indicators which would establish when nonlinearity is significant.

(4) A statistical response model should be developed using the statistics of the eigenvectors.

(5) The assumption of statistical independence of adjoining beam elements and adjoining plate elements is convenient but not always valid. A model should be developed to permit statistical correlation between the properties of adjoining elements.

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APPENDIX A

NOMENCLATURE

A.1 EIGENVALUE-EIGENVECTOR PARTIAL DERIVATIVE DEVELOPMENT DEFINITIONS

x_i - i^{th} eigenvector

x_{ui} - u^{th} component of i^{th} eigenvector equivalent to x(i,u) in computer output

λ_i - i^{th} eigenvalue

K or $[K]$ - stiffness matrix

k_{rs} - $(rs)^{th}$ component of the stiffness matrix

M or $[M]$ - mass matrix

m_{rs} - $(rs)^{th}$ component of the mass matrix

M_i - generalized mass; $x_i{}'Mx_i$, a scalar quantity

$\left\{\overline{Mx_i}^{u}\right\}$ - vector formed from the product Mx_i with the u^{th} element removed

F_i or $[F_i]$ - the matrix, $K-\lambda_i M$

$\left[\overline{F_i}^{u}\right]$ - the matrix F_i reduced by removal of the u^{th} row and u^{th} column.

$\left\{\overline{\delta_{jr}}^{u}\right\}$ - vector of length n-1 with zero elements in every location except r where the element is equal to 1. If u is less than r, the location is at r-1; and if u=r, the vector is a zero vector.

$\left\{ \frac{\partial x_i}{\partial k_{rs}} \right\}$ - a vector of partial derivatives of the x_i eigenvector with respect to the k_{rs} element in the stiffness matrix.

$\frac{\partial x_{ui}}{\partial k_{rs}}$ - the u^{th} element of the vector $\left\{ \frac{\partial x_i}{\partial k_{rs}} \right\}$

$\left\{ \overline{\frac{\partial x_i}{\partial k_{rs}}}^{u} \right\}$ - the vector $\left\{ \frac{\partial x_i}{\partial k_{rs}} \right\}$ reduced by removal of the element, $\frac{\partial x_{ui}}{\partial k_{rs}}$

A.2 COORDINATE DEFINITIONS (SEE FIGURES 4-1 and 4-4)

X_m, Y_m, Z_m - structural member axes

X_s, Y_s, Z_s - system coordinate orientation

u_i, v_i, w_i - deflections in the member X_m, Y_m, Z_m directions respectively at the i^{th} node.

$\Theta_{x_i}, \Theta_{y_i}$ - rotations about the member X_m, Y_m axes respectively at the i^{th} node

A.3 GENERAL MATRIX AND VECTOR DEFINITIONS

$[\quad]$ - matrix

$[\quad]'$ - transpose of a matrix

$\{\quad\}$ - vector

$\left[\overline{\quad}^{u}\right]$ - a matrix reduced in size by removal of the u^{th} row and u^{th} column

$\left\{ \overline{\quad}^{u} \right\}$ - a vector reduced in size by removal of the u^{th} element.

$\left\{ \quad ^{u} \right\}$ - a vector with the u^{th} element set equal to zero

$[S]$ - an upper triangular matrix used in a decomposition operation

$[D]$ - a diagonal matrix used in a decomposition operation

$[KR]$, $[KS]$ - matrices of indices used to develop sets of compatible partial derivatives (see Section 6.2)

A.4 STATISTICAL DEFINITIONS

$(\overline{\quad})$ - mean value of ( )

σ - standard deviation

Cov (x_i, x_j) - covariance of x_i and x_j

ρ_{ij} - correlation coefficients between variables x_i and x_j

$$\rho_{ij} = \frac{\mathrm{Cov}(x_i, x_j)}{\sigma_{x_i} \sigma_{x_j}}$$

A.5 DEFINITIONS USED IN THE DEVELOPMENT OF BEAM AND PLATE STIFFNESS MATRICES

Beam

E - Young's modulus of elasticity

A - Cross-sectional area of a beam element

L - length of the beam element

G - modulus of rigidity

J - beam cross-sectional polar moment of inertia about the X_m axis

ν - Poisson's ratio

I_2, I_3 - beam cross-sectional moments of inertia about Y_m and Z_m axes respectively

SF_3, SF_2 - beam cross-sectional shear factors about the Y_m and Z_m axes respectively

$$\Phi_2 = \frac{12\ EI_3}{GA\ SF_3\ L^2}$$

$$\Phi_3 = \frac{12\ EI_2}{GA\ SF_2\ L^2}$$

$[\gamma]$ - coordinate rotation matrix (3 x 3)

$[R]$ - rotation matrix for a beam (12 x 12)

p_j - j^{th} physical property

Plate

A - surface area (one side of the plate)

$L_{\alpha\beta}$ - distance between nodes α and β

$x_{\alpha\beta}$, $y_{\alpha\beta}$ - x and y distances between nodes α and β in X_m, Y_m, Z_m coordinates

T_p - total thickness of the plate

T_c - thickness of the core of the plate

A_w - effective shear web area

G - modulus of rigidity of the core

ν - Poisson's ratio of the face sheets

E - Young's modulus of the face sheets

p_j - j^{th} physical property

$[K_b]$ - stiffness matrix for plate bending matrix

$[K_n]$ - plate stiffness matrix due to normal stresses

$[K_s]$ - plate stiffness matrix due to out-of-plane shear

$[K_w]$ - plate stiffness matrix due to in-plane shear

$[K_1], [K_2], [K_3], [K_4]$ - component matrices of the plate stiffness matrix

$[R_{plate}]$ - rotation matrix for a plate (15 x 18)

$$\psi = \frac{1}{2(1-\nu^2)A}$$

$$\lambda_1 = \frac{1-\nu}{2}$$

$$\lambda_2 = \frac{1+\nu}{2}$$

APPENDIX B

USER'S MANUAL

B.1 INTRODUCTION

VIDAP is written in seven sections consisting of a main control program and six overlays. These sections are described briefly below:

OVERLAY (VIDAP, 0, 0) is the main program and reads in the run identification, matrix dimension, matrix bandwidth, control flags, node constraints, user's output selection, and calls the various overlays as needed.

OVERLAY (VIDAP, 1, 0) is used to load the mass matrix, stiffness matrix, eigenvalues, and eigenvectors onto magnetic tape for use by other OVERLAY sections.

OVERLAY (VIDAP, 2, 0) reads in the geometry data or the matrix of partial derivatives of stiffness and mass with respect to properties, $\left[{}^{(i)}\frac{\partial(k,m)}{\partial(p)}\right]$. If the geometry data is entered, it computes $\left[{}^{(i)}\frac{\partial(k,m)}{\partial(p)}\right]$. In either case $\left[{}^{(i)}\frac{\partial(k,m)}{\partial(p)}\right]$ is stored on tape. The covariance matrix is also entered by this section of the program.

OVERLAY (VIDAP, 3, 0) determines the location of the related elements in the stiffness matrix for the selected section of the model under investigation from the constraint and node data. If $\left[{}^{(i)}\frac{\partial(k,m)}{\partial(p)}\right]$ is entered, the element positions in the stiffness matrix are entered by this section.

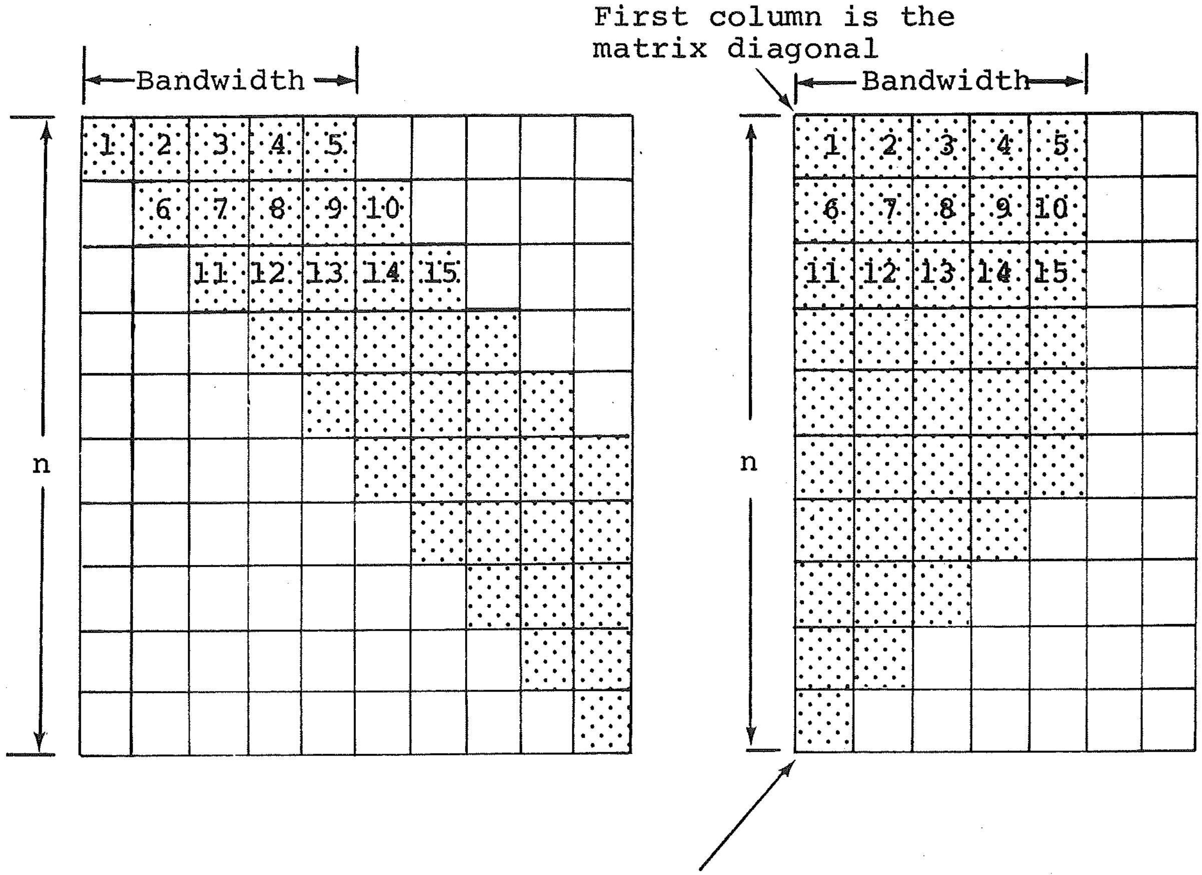
OVERLAY (VIDAP, 4, 0) computes the partial derivatives of the eigenvalues-vectors with respect to stiffness and mass, $\left[\frac{\partial(\lambda,x)}{\partial(k_r,m)}\right]$, stores them on tape for later processing. These partials may be printed as output if desired.

OVERLAY (VIDAP, 5, 0) computes the product of the partial derivative matrices and the covariance matrix for each structural element and sums them in preparation for computing the final answers. This section outputs the partial derivatives if requested.

OVERLAY (VIDAP, 6, 0) computes the correlation matrix and standard deviations and outputs the answers.

MASS AND STIFFNESS MATRIX

The mass and stiffness matrices must be symmetrical, with a maximum bandwidth of 80 and a maximum dimension of 300. They are entered in a rectangular configuration as shown below:



Data entered & stored in this configuration

CONTROL FLAGS

The control flags give the options of: (1) entering the partial derivatives of stiffness and mass with respect to properties or computing these partials, (2) printing-out the partial derivatives of eigenvalues-eigenvectors with respect to stiffness and mass and partial derivatives of stiffness and mass with respect to properties.

CONSTRAINTS

The node number and constraints must be entered for each node having constraints. The node numbers may vary from 1 to 100 with a maximum of 100 nodes. Each node has six degrees of freedom unless constrained. (NOTE: the program can handle a maximum of 300 d.o.f.)

OUTPUT SELECTION

A maximum of 100 elements may be selected for the output. This may be a combination of eigenvalues and selected elements from the associated eigenvectors, or eigenvalues alone. The program checks to see if the mode (1 to 300) has been requested. If so, the associated eigenvalue becomes one of the output elements. The program will then check the two start and stop numbers for the associated eigenvector. If the first start number is anything other than 0, it will determine the start and stop element for the first string of eigenvector elements associated with this mode. It will then check the second start; and if it is not 0, it will determine the start and stop elements for the second string of eigenvector elements associated with this mode.

Example:

Statistics are desired for the second mode (second eigenvalue) and elements 1 through 4 and 23 through 24. (Note that only two strings of contiguous elements are allowed per mode).

The output would give:

Mode	Eigenvalue	Standard Deviation	Freq.	S.D.
2	xx	xx	xx	xx

Vector	Eigenvector	Standard Deviation
x(2,1)	xx	xx
x(2,2)	xx	xx
x(2,3)	xx	xx
x(2,4)	xx	xx
x(2,23)	xx	xx
x(2,24)	xx	xx

INPUT-OUTPUT TAPES

Logical 1, 2, 3, 4, 9, and 10 are the Fortran tape units required for scratch tapes (binary format). Logical 5 is the input (card reader) and Logical 6 is the output (printer).

B.2 CARD PREPARATION

1. Case Heading (16A5)

   This card may contain any alphanumeric character in columns 1-80 for problem identification.

2. N, KBW (2I8)

1 – 8	9 – 16	
N	KBW	

<u>Column 1-8</u> - Integer must be right justified

N = mass and stiffness matrix dimension, eigenvector length

<u>Column 9-16</u> - Integer must be right justified

KBW = mass and stiffness matrix bandwidth dimension

3. IFLAG JFLAG (2I8)

1 – 8	9 – 16	
IFLAG	JFLAG	

<u>Column 1-8</u> - Integer must be right justified

IFLAG = 0 = no printout of partial derivatives
IFLAG = 1 = printout of all partial derivatives

<u>Column 9-16</u> - Integer must be right justified

JFLAG = 0 = compute partial derivatives with respect to property
JFLAG = 1 = input partial derivatives with respect to property

4. LOT (I8)

1 – 8	
LOT	

<u>Column 1-8</u> - Integer must be right justified

LOT = node number of node with constraints (1 to 100 acceptable)
LOT = 0 or negative number is a sentinal to denote that all constraints have been entered

5. KONSTR (1),...,KONSTR (6), (6I8), (For above node number)

1 ... 8	9 ... 16	17 ... 24	25 ... 32	33 ... 40	41 ... 48	
KONSTR 1	KONSTR 2	KONSTR 3	KONSTR 4	KONSTR 5	KONSTR 6	

Column 1-8 - Integer must be right justified

KONSTR = 0 = no constraint for first d.o.f.*
KONSTR = 1 = constraint for first degree-of-freedom

Column 9-16 - Integer must be right justified

KONSTR = 0 = no constraint for second d.o.f.
KONSTR = 1 = constraint for second d.o.f.

Column 17-24 - Integer must be right justified

KONSTR = 0 = no constraint for third d.o.f.
KONSTR = 1 = constraint for third d.o.f.

Column 25-32 - Integer must be right justified

KONSTR = 0 = no constraint for fourth d.o.f.
KONSTR = 1 = constraint for fourth d.o.f.

Column 33-40 - Integer must be right justified

KONSTR = 0 = no constraint for fifth d.o.f.
KONSTR = 1 = constraint for fifth d.o.f.

Column 41-48 - Integer must be right justified

KONSTR = 0 = no constraint for sixth d.o.f.
KONSTR = 1 = constraint for sixth d.o.f.

NOTE: Only LOT (Node Number) and constraints for LOT need to be entered for those nodes with constraints. The program assumes that the only nodes with constraints are those where constraint data is entered. If LOT is zero or negative, this signals the program that all constraints have been entered. Example: if the system has no constraints, only one (1) card would be required: LOT = zero or negative. LOT (Nodes) must not exceed one hundred (100) in number or level.

*The order of the six degrees of freedom is as shown in Figure 4-1

6. KODE, KK (2I8) (Output select cards)

1 ... 8	9 ... 16	
KODE	KK	

Column 1-8 - Integer must be right justified

KODE = 0 = continue to read select values
KODE = -1 = all selected mode numbers entered

Column 9-16 - Integer must be right justified

KK = number of mode selected; i.e., 1 = first mode, 2 = second mode, etc. KK may vary from 1 to N (Number of degrees-of-freedom of the system). Maximum d.o.f. acceptable by the program is 300.

7. LGO (1), LGO (2), LSTOP (1), LSTOP (2), (4I8)

1 — 8	9 — 16	17 — 24	25 — 32	
LGO (1)	LGO (2)	LSTOP (1)	LSTOP (2)	

Column 1-8 - Integer must be right justified

LGO (1) = number of element for start of first selected section of vector (x)

Column 9-16 - Integer must be right justified

LGO (2) = number of element for start of second selected section of vector (x)

Column 17-24 - Integer must be right justified

LSTOP (1) = number of element for last value of first selected section of vector (x)

Column 25-32 - Integer must be right justified

LSTOP (2) = number of element for last value of second selected section of vector (x)

The output select cards establish the amount and order of printout of the eigenvalue and eigenvector statistics as shown in the introduction of the appendix. The output will always label the statistical data but does not label the partial derivatives which are listed in the output as "PARTIAL DERIVATIVES FOR THE EIGENVALUES-EIGENVECTORS (ELEMENT ___)." The order of these partial derivatives in the printout are established by LGO (1), LGO (2), etc. and an example of the output is shown below.

Example: LGO (1) = 3 LGO (2) = 12
LSTOP (1) = 9 LSTOP (2) = 15

The following data would be saved for computation of statistics from the partial derivative matrix (for Mode 5):

KK	$\frac{\partial\lambda_5}{\partial k_{11}}$	$\frac{\partial\lambda_5}{\partial k_{12}}$	....	$\frac{\partial\lambda_5}{m_{11}}$	$\frac{\partial\lambda_5}{\partial m_{22}}$	..	
LGO (1)	$\frac{\partial x_{35}}{\partial k_{11}}$	$\frac{\partial x_{35}}{k_{12}}$	....	$\frac{\partial x_{35}}{m_{11}}$	$\frac{\partial x_{35}}{\partial_{22}}$	..	
	.	.		.	.		partials for consect. eigen-
	.	.		.	.		vector elements x_{35} through x_{95}
	.	.		.	.		
LSTOP (1)	$\frac{\partial x_{95}}{k_{11}}$	$\frac{\partial x_{95}}{k_{12}}$	....	$\frac{\partial x_{95}}{m_{11}}$	$\frac{\partial x_{95}}{\partial m_{22}}$	..	
LGO (2)	$\frac{\partial x_{12,5}}{k_{11}}$	$\frac{\partial x_{12,5}}{k_{12}}$	....	$\frac{\partial x_{12,5}}{m_{11}}$	$\frac{\partial x_{12,5}}{\partial m_{22}}$	..	
	.	.		.	.		partials for consect. eigen-
	.	.		.	.		vector elements $x_{12,5}$ thru $x_{15,5}$
	.	.		.	.		
LSTOP (2)	$\frac{\partial x_{15,5}}{\partial k_{11}}$	$\frac{\partial x_{15,5}}{\partial k_{12}}$	....	$\frac{\partial x_{15,5}}{\partial m_{11}}$	$\frac{\partial x_{15,5}}{\partial m_{22}}$	..	

If both LGO's = 0, only the partial derivative of the eigenvalue will be computed and saved (KK).

Once the partials for an eigenvalue and associated eigenvector elements have been listed, the procedure repeats for the next eigenvalue and eigenvector elements as specified by the output select cards.

The order of printout discussed above and demonstrated in the example is also used in the format of the correlation matrix. To demonstrate this consider a very simple output selection such as that below.

Sample Output Select Cards

KODE

0	1	

LGO (1), LGO (2), LSTOP (1), LSTOP (2)

1	4	2	5	

KODE

0	2	

LGO (1), LGO (2), LSTOP (1), LSTOP (2)

2	5	3	6	

KODE

-1	

The resulting correlation matrix will have the dimension of 10 x 10 with the elements down the column and across the rows in the following order:

$$\begin{array}{c c}
 & \begin{array}{cccccccccc} \lambda_1 & x_{11} & x_{21} & x_{41} & x_{51} & \lambda_2 & x_{22} & x_{32} & x_{52} & x_{62} \end{array} \\
\begin{array}{c} \lambda_1 \\ x_{11} \\ x_{21} \\ x_{41} \\ x_{51} \\ \lambda_2 \\ x_{22} \\ x_{32} \\ x_{52} \\ x_{62} \end{array} &
\left[\begin{array}{cccccccccc}
1 & \cdot & & & & & & & & \\
\cdot & 1 & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
\multicolumn{10}{c}{\text{Correlation Matrix}} \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & & \\
 & & & & & & & & 1 & \cdot \\
 & & & & & & & & \cdot & 1
\end{array}\right]
\end{array}$$

Note that the eigenvector components contain the mode number in the second subscript. This order is reversed in the final output. Thus,

eigenvector element number

$$x\ (3,\ 2) \equiv x_{23}$$

mode number

8. M (Mass Matrix Data) (5E16.8)

1	13	16 17	29	32 33	45	48 49	61	64 65	77	80
K	E		SK	E		SK	E		SK	E		SK	E	

The mass matrix* is entered in the E Format with decimal point between columns 4 and 5, 20 and 21, 36 and 37, 52 and 53, and 68 and 69.

Each new row in the matrix must start on a new card.

Example: KBW = 8 (Bandwidth of 8)

Card 1 = 5 values M_{11}, M_{12}, M_{13}, M_{14}, M_{15}

Card 2 = 3 values M_{16}, M_{17}, M_{18}, Blank, Blank

Card 3 = 5 values M_{22}, M_{23}, M_{24}, M_{25}, M_{26}

*The mass matrix must be in consistent units, the geometry data in inches, the stiffness matrix in in-lbs, and the mass will be pounds weight.

9. K (Stiffness Matrix Data) (5E16.8)

Same as Mass Matrix Setup

10. LAMBDA, CODE (2E16.8)*

1 13 16	17 29 32	33
E	E	

Same Format as 8 and 9

LAMBDA = Eigenvalue

CODE = 0 = continue

CODE = -1 = No further modes to enter

*See 11. on next page

11. (X) (Eigenvector) (5E16.8)*

Same format and rules as used in 8 and 9

*Data in steps 10 and 11 are entered in pairs; i.e., the eigenvalue and the associated eigenvector are paired together sequentially. No eigenvalue can be entered without an eigenvector and vice versa.

12. WEIG1(I), WEIG2(I), WEIG3(I), THE1(I), THE2(I), THE3(I) (6F8.0)

1 8	9 16	17 24	25 32	33 40	41 48	
WEIG1(I)	WEIG2(I)	WEIG3(I)	THE1(I)	THE2(I)	THE3(I)	

3 cards required - arranged in ascending node order
Decimal point right justified unless entered

Column 1-8 **

WEIG1(I) = Weight associated with first degree-of-freedom of selected element for the (I)th node.

Column 9-16

WEIG2(I) = Weight associated with second d.o.f.

Column 17-24

WEIG3(I) = Weight associated with third d.o.f.

Column 25-32

THE1(I) = Moment of Inertia associated with fourth d.o.f.

Column 33-40

THE2(I) = Moment of Inertia associated with fifth d.o.f.

Column 41-48

THE3(I) = Moment of Inertia associated with sixth d.o.f.

**Weights are entered in pounds and divided by 386.4 within the program. Moments of inertia must be entered in terms of mass-in^2 where mass is weight divided by 386.4 in/sec^2.

13. COV (5E16.8) (See examples in Sections 5 and 7 for covariance matrix development)

Same Format as 8

Covariance Matrix (9 x 9)

First card: COV_{11}, COV_{12}, COV_{13}, COV_{14}, COV_{15}
Second card: COV_{16}, COV_{17}, COV_{18}, COV_{19}
Third card: COV_{21}, COV_{22}, COV_{23}, COV_{24}, COV_{25}
et cetera . . . . .

. . . . . .

Eighteenth card: COV_{96}, COV_{97}, COV_{98}, COV_{99},

14. JNODE, ITYPE, X1, X2, X3 (2I8, 3F8.0) (3 cards (nodes) required)

1 8	9 16	17 24	25 32	33 40
JNODE	ITYPE	X1	X2	X3

Column 1-8 - Integer must be right justified

JNODE = Node ID number (Node numbers must be in ascending order, i.e. JNODE of card 1 < JNODE card 2 < JNODE card 3)

Column 9-16 - Integer must be right justified

ITYPE = 1 - Beam Element
ITYPE = 2 - Plate Element

Column 17-24 - Decimal right justified unless punched

X1 = Coordinate value of X for JNODE

Column 25-32 - Decimal right justified unless punched

X2 = Coordinate value of Y for JNODE

Column 33-40 - Decimal right justified unless punched

X3 = Coordinate value of Z for JNODE

15a. Bar Element

E, PR, AREA, XJ, XI1, XI2, SF2, SF3 (8F8.0)

1 8	9 16	17 24	25 32	33 40	41 48	49 56	57 64	
E	PR	AREA	XJ	XI1	XI2	SF2	SF3	

Column 1-8 - Decimal right justified unless punched.

E = Modulus of elasticity for bar member.

Column 9-16 - Decimal right justified unless punched.

PR = Poisson's ratio for bar element.

Column 17-24 - Decimal right justified unless punched.

AREA = Cross sectional area of bar.

Column 25-32 - Decimal right justified unless punched.

XJ* = Bar member torsional constant.

---

*See Section B.4 for table of torsional constants

Column 33-40 - Decimal right justified unless punched.

XI1 = Moment of inertia about the Y_m axis (Fig. 4-1).

Column 41-48 - Decimal right justified unless punched.

XI2 = Moment of inertia about the Z_m axis (Fig. 4-1).

Column 49-56 - Decimal right justified unless punched.

SF2* = Shear shape factor 2.

Column 57-64 - Decimal right justified unless punched.

SF3* = Shear shape factor 3.

---

* Shear shape factors for bending about the Y_m and Z_m axes are given in Section 4. These factors, when multiplied by the bar cross-section area, will yield effective shear areas for the member. The permissible range of SF2 and SF3 is:

$$.01 \leqq SF2 \text{ of } SF3 \leqq 999.99$$

15b. Plate Element

J1, J2, J3, TP, E, PR, AW, G (3I8, SF8.0)

1 8	9 16	17 24	25 32	33 40	41 48	49 56	57 64
J1	J2	J3	TP	E	PR	AW	G

Column 1-8 - Integer must be right justified

J1 = First node of plate (nodes must appear in ascending order (i.e. J1 < J2 < J3)

Column 9-16 - Integer must be right justified

J2 = Second node of plate.

Column 17-24 - Integer must be right justified

J3 = Third node of plate

Column 25-32 - Decimal right justified unless punched

TP = Total thickness of plate

Column 33-40 - Decimal right justified unless punched

E = Modulus of elasticity for face sheets

Column 41-48 - Decimal right justified unless punched

PR = Poisson's ratio

Column 49-56 - Decimal right justified unless punched

Core thickness in inches

Column 57-64 - Decimal right justified unless punched

Shear Modulus of core

16a. Bar Element

KEY (1), KEY (2) . . . . KEY (8) (8I8)

1 8	9 16	17 24	25 32	33 40	41 48	49 56	57 64	
KEY(1)	KEY(2)	KEY(3)	KEY(4)	KEY(5)	KEY(6)	KEY(7)	KEY(8)	

Column 1-8 - Integer must be right justified

KEY(1) = Selected integer from 1 to 8

Column 9-16 - Integer must be right justified

KEY(2) = Selected integer from 1 to 8

et cetera

The keys identify the independent parameters to be random. For a bar element, the code is as follows:

KEY = 1 Random modulus of elasticity
KEY = 2 Random cross-sectional area
KEY = 3 Random moment of inertia about the Y_m axis
KEY = 4 Random moment of inertia about the Z_m axis
KEY = 5 Random Poisson's ratio
KEY = 6 Random shear shape factor for bending about the Y_m axis
KEY = 7 Random shear shape factor for bending about the Z_m axis
KEY = 8 Random bar member torsional constant

Note: Keys do not have to be entered in sequence.

Example: Desire Random 1, 4 and 6

1 – 8	9 – 16	17 – 24	25 –
1	6	4	

16b. Plate Element

KEY (1), KEY (2), . . . KEY (5) (5I8)

1 – 8	9 – 16	17 – 24	25 – 32	33 – 40	
KEY(1)	KEY(2)	KEY(3)	KEY(4)	KEY(5)	

<u>Column 1-8</u> - Integer must be right justified

KEY(1) - Selected integer from 1 to 5

et cetera

KEY = 1 Random shear modulus
KEY = 2 Random shear web cross-sectional area
KEY = 3 Random thickness of plate
KEY = 4 Random modulus of elasticity
KEY = 5 Random Poisson's ratio

Note: Keys do not have to be entered in sequence.
See example in 16a.

17. KIN (I8) (Only when JFLAG = 1 and partial derivatives are entered into the program)

1 – 8	
KIN	

<u>Column 1-8</u> - Integer must be right justified

KIN = Number of rows in the partial derivative matrix to be entered
Note KIN may not exceed 189

18. A (I,J) (10F8.0) Partial derivative matrix data

1 – 8	9 – 16	17 – 24	25 – 32	33 – 40	41 – 48	49 – 56	57 – 64	65 – 72	73 – 80
A(I,J)									

<u>Column 1-8</u> - Decimal point right justified unless punched

Note: Number of cards A(I,J) must equal KIN, J may not exceed 9.

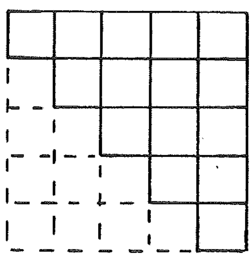
19. LKRS (I8) (Only when JFLAG = 1)

1 – 8	
LKRS	

<u>Column 1-8</u> - Integer must be right justified

LKRS = Dimension of the selected KR and KS matrices

LKRS



LKRS

LKRS may not exceed 10

20. KR (10I8)

1 – 8	9 – 16	17
		etc.

Matrix with row number for each element

All KR = KS (i.e. diagonal stiffness matrix elements) must be on the diagonal only
All KR ≠ KS must be in off diagonal locations only

21. KS (10I8)

1 8 9 16 17

Matrix with column number for each element

The upper triangular section will determine which partial derivatives will be computed.

Example

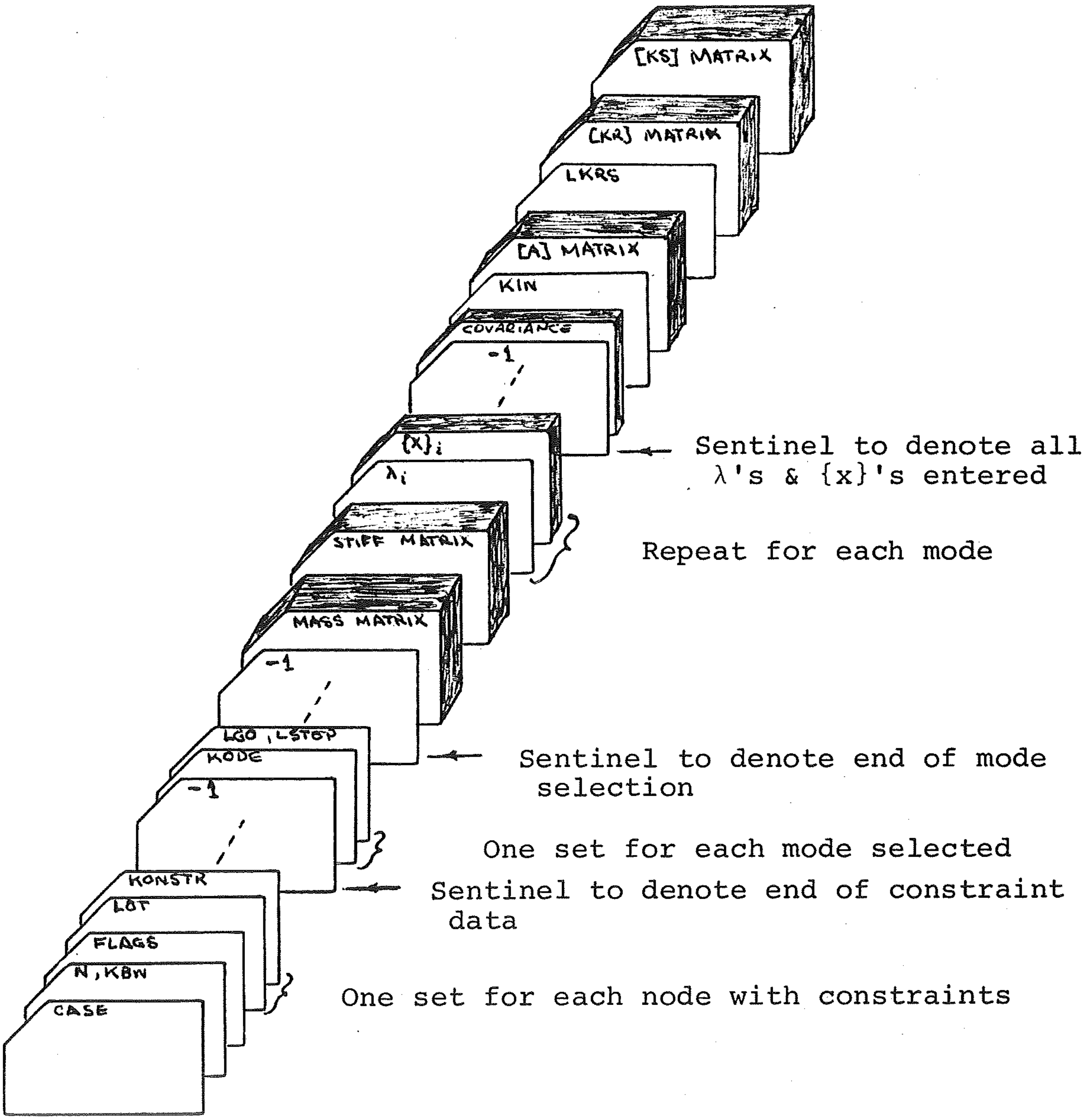
$$KR = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \qquad KS = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

The partial derivatives for each row would be ordered in the printout as follows:

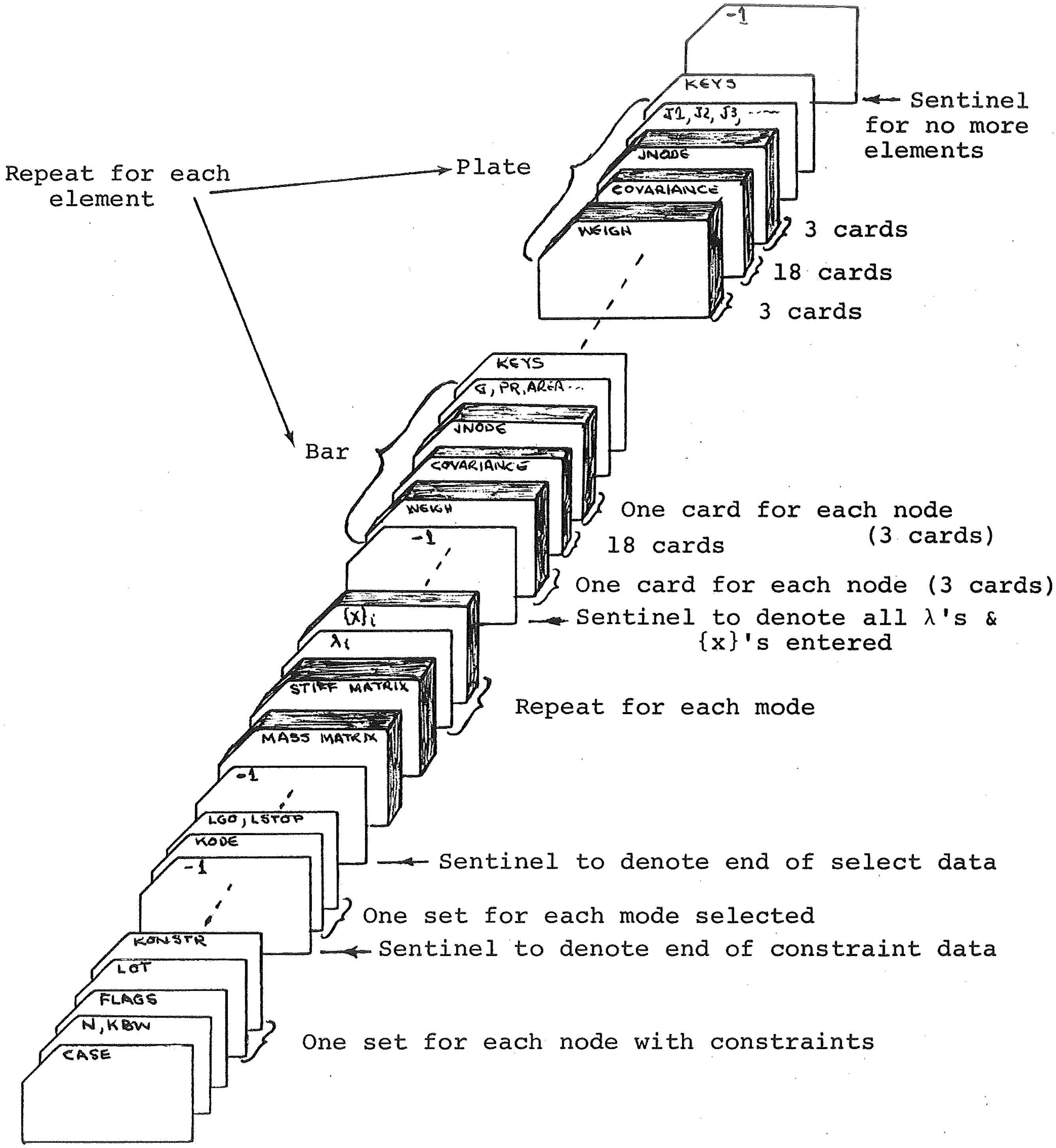
$$\frac{\partial}{\partial K_{(KR_{(1,1)})(KS_{(1,1)})}}, \frac{\partial}{\partial K_{(KR_{(1,2)})(KS_{(1,2)})}} \cdots$$

followed by $\partial/\partial m$; for example

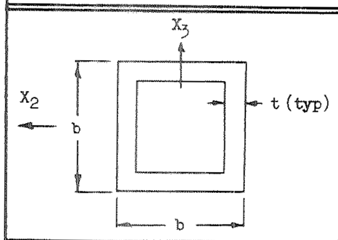
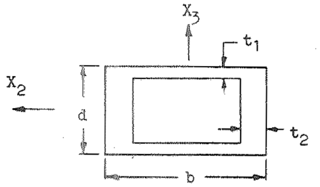
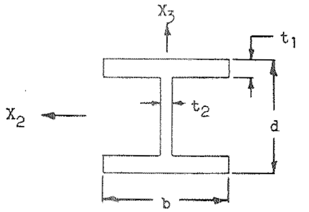
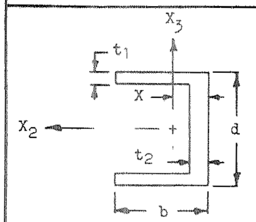
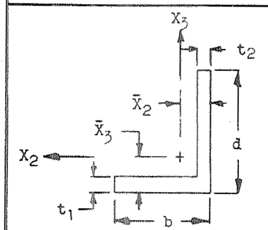
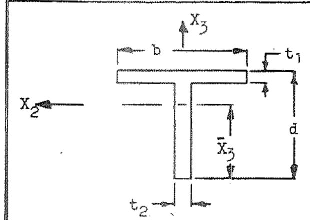
$$\frac{\partial}{\partial k_{11}}, \frac{\partial}{\partial k_{12}}, \frac{\partial}{\partial k_{13}}, \frac{\partial}{\partial k_{22}}, \frac{\partial}{\partial k_{23}}, \frac{\partial}{\partial k_{33}}, \frac{\partial}{\partial m_{11}}, \frac{\partial}{\partial m_{22}}, \frac{\partial}{\partial m_{33}}$$



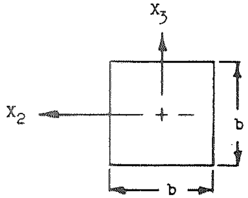
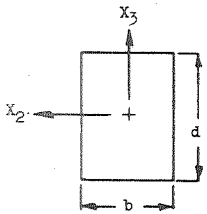
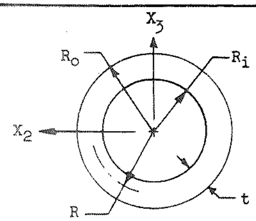
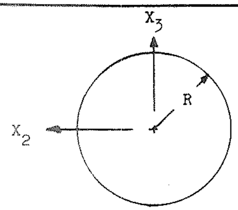
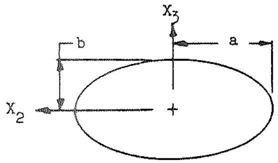
Case I (JFLAG = 1)



Case II (JFLAG = 0)

CONFIGURATION	AREA, A	TORSIONAL CONSTANT, J	MOMENTS OF INERTIA, I_2 AND I_3	K_2	K_3
	$A = b^2 - (b-2t)^2$	$J = (b-t)^3 t$	$I_2 = I_3 = \frac{1}{12}\left[b^4 - (b-2t)^4\right]$	$b_2 = 2t$ $Q_2 = \frac{b^3 - (b-2t)^3}{8}$ $K_2 = \frac{I_2 b_2}{Q_2 A}$	b_2 = same Q_3 = same K_3 = same
	$A = bd - (b-2t_2)(d-2t_1)$	$J = \left[\frac{2t_1t_2(b-t_2)^2(d-t_1)^2}{t_2(b-t_2) + t_1(d-t_1)}\right]$	$I_2 = \frac{1}{12}\left[bd^3 - (b-2t_2)(d-2t_1)^3\right]$ $I_3 = \frac{1}{12}\left[db^3 - (d-2t_1)(b-2t_2)^3\right]$	$b_2 = 2t_2$ $Q_2 = \left[\frac{bt_1(d-t_1)}{2} + \frac{t_2(d-2t_1)^2}{4}\right]$ $K_2 = \frac{I_2 b_2}{Q_2 A}$	$b_3 = 2t_1$ $Q_3 = \left[\frac{dt_2(b-t_2)}{2} + \frac{t_1(b-2t_2)^2}{4}\right]$ $K_3 = \frac{I_3 b_3}{Q_3 A}$
	$A = 2bt_1 + t_2(d-2t_1)$	$J = 2\beta_1 bt_1^3 + \beta_2(d-2t_1)t_2^3$ [See Fig. 1 for β]	$I_2 = \frac{1}{12}\left[bd^3 - (b-t_2)(d-2t_1)^3\right]$ $I_3 = \frac{1}{12}\left[2t_1b^3 + (d-2t_1)t_2^3\right]$	$b_2 = t_2$ $Q_2 = \left[\frac{bt_1(d-t_1)}{2} + \frac{t_2(d-2t_1)^2}{8}\right]$ $K_2 = \frac{I_2 b_2}{Q_2 A}$	$b_3 = 2t_1$ $Q_3 = \frac{t_1(b^2 - t_2^2)}{4}$ $K_3 = \frac{I_3 b_3}{Q_3 A}$
	$A = bd - (b-t_2)(d-2t_1)$ $X = \frac{2b^2t_1 + (d-2t_1)t_2^2}{2\left[2bt_1 + (d-2t_1)t_2\right]}$	$J = \beta_1 dt_2^3 + 2\beta_2(b-t_2)t_1^3$ [See Fig. 1 for β]	$I_2 = \frac{1}{12}\left[bd^3 - (b-t_2)(d-2t_1)^3\right]$ $I_3 = \frac{2t_1b^3}{12} + \frac{(d-2t_1)t_2^3}{12}$ $+ 2bt_1\left(\frac{b}{2} - X\right)^2 + t_2(d-2t_1)\left(X - \frac{t_2}{2}\right)^2$	$b_2 = t_2$ $Q_2 = \left[\frac{t_2(d-2t_1)^2}{8} + \frac{bt_1(d-t_1)}{2}\right]$ $K_2 = \frac{I_2 b_2}{Q_2 A}$	$b_3 = 2t_1$ $Q_3 = t_1(b-X)^2$ $K_3 = \frac{I_3 b_3}{Q_3 A}$
	$A = dt_2 + (b-t_2)t_1$ $\bar{X}_2 = \frac{\frac{b^2t_1}{2} + \frac{(d-t_1)t_2^2}{2}}{A}$ $\bar{X}_3 = \frac{\frac{d^2t_2}{2} + \frac{(b-t_2)t_1^2}{2}}{A}$	$J = \beta_1 dt_2^3 + \beta_2(b-t_2)t_1^3$	$I_2 = \frac{t_2d^3}{12} + \frac{(b-t_2)t_1^3}{12}$ $+ dt_2\left(\frac{d}{2} - \bar{X}_3\right)^2 + (b-t_2)t_1\left(\bar{X}_3 - \frac{t_1}{2}\right)^2$ $I_3 = \frac{t_1b^3}{12} + \frac{(d-t_1)t_2^3}{12}$ $+ bt_1\left(\frac{b}{2} - \bar{X}_2\right)^2 + (d-t_1)t_2\left(\bar{X}_2 - \frac{t_2}{2}\right)^2$	$b_2 = t_2$ $Q_2 = \frac{t_2(d-\bar{X}_3)^2}{2}$ $K_2 = \frac{I_2 b_2}{Q_2 A}$	$b_3 = t_1$ $Q_3 = \frac{t_1(b-\bar{X}_2)^2}{2}$ $K_3 = \frac{I_3 b_3}{Q_3 A}$
	$A = bt_1 + (d-t_1)t_2$ $X_3 = \frac{bt_1\left(d - \frac{t_1}{2}\right) + \frac{(d-t_1)^2t_2}{2}}{A}$	$J = \beta_1 bt_1^3 + \beta_2(d-t_1)t_2^3$	$I_2 = \frac{bt_1^3}{12} + \frac{t_2(d-t_1)^3}{12}$ $+ bt_1\left(d - \bar{X}_3 - \frac{t_1}{2}\right)^2$ $+ (d-t_1)t_2\left(\bar{X}_3 - \frac{d_1 - t_1}{2}\right)^2$ $I_3 = \frac{t_1b^3}{12} + \frac{(d-t_1)t_2^3}{12}$	$b_2 = t_2$ $Q_2 = \frac{\bar{X}_3^2 t_2}{2}$ $K_2 = \frac{I_2 b_2}{Q_2 A}$	$b_3 = t_1$ $Q_3 = \frac{t_1(b^2 - t_2^2)}{8}$ $K_3 = \frac{I_3 b_3}{Q_3 A}$

*Reference: Rosen, Richard, Stardyne User's Manual
Mechanics Research, Inc., Los Angeles, March 1970

CONFIGURATION	AREA, A	TORSIONAL CONSTANT, J	MOMENTS OF INERTIA, I_2 AND I_3	SHEAR COEFFICIENT K_2	SHEAR COEFFICIENT K_3
	$A = b^2$	$J = 0.1406\, b^4$	$I_2 = I_3 = \frac{b^4}{12}$	$K_2 = 0.67$	$K_3 = 0.67$
	$A = bd$	$J = db^3\left[\frac{1}{3} - 0.21\,\frac{b}{d}\left(1 - \frac{b^4}{12d^4}\right)\right]$	$I_2 = \frac{bd^3}{12}$ $I_3 = \frac{db^3}{12}$	$K_2 = 0.67$	$K_3 = 0.67$
	Thick wall: $A = \pi\left[R_o^2 - R_i^2\right]$ Thin wall: $A = 2\pi Rt$	Thick wall: $J = \frac{\pi}{2}\left[R_o^4 - R_i^4\right]$ Thin wall: $J = 2\pi R^3 t$	Thick wall: $I_2 = I_3 = \frac{\pi}{4}\left[R_o^4 - R_i^4\right]$ Thin wall: $I_2 = I_3 = \pi R^3 t$	Thick wall: $K_2 = K_3 = \frac{\left(R_o^4 - R_i^4\right)\left(R_o - R_i\right)}{0.4244\pi\left(R_o^3 - R_i^3\right)\left(R_o^2 - R_i^2\right)}$ Thin wall: $K_2 = K_3 = 0.5$	
	$A = \pi R^2$	$J = I_2 + I_3 = \frac{\pi R^4}{2}$	$I_2 = I_3 = \frac{\pi R^4}{4}$	$K_2 = 0.75$	$K_3 = 0.75$
	$A = \pi ab$	$J = \frac{\pi a^3 b^3}{a^2 + b^2}$	$I_2 = \frac{\pi ab^3}{4}$ $I_3 = \frac{\pi ba^3}{4}$	$K_2 = 0.75$	$K_3 = 0.75$